Weakly Compatible Maps in Complex Valued G-Metric Spaces

S. Mudgal
Department of Mathematics
Echelon Institute of Technology
Jasana, Faridabad:121101, India

ABSTRACT
In this paper, we introduce the notion of complex valued G-metric spaces and prove a common fixed point theorem for weakly compatible maps in this newly defined spaces.

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1. INTRODUCTION
The study of fixed points of mappings satisfying certain contractive conditions have been at the center of rigorous research activity. Recently, Mustafa and Sims [8,9] have shown that most of the results concerning Dhage’s D-metric spaces are invalid, therefore they introduced an improved version of the generalized metric space structure which they called G-metric spaces.

In 2006, Mustafa and Sims [9] introduced the concept of G-metric spaces as follows:

Definition 1.1. Let X be a non-empty set, and let G: \( X \times X \times X \rightarrow \mathbb{R}^+ \) be a function satisfying the following properties:

(G1) \( G(x, y, z) = 0 \) if \( x = y = z \),
(G2) \( 0 < G(x, y, z) \) for all \( x, y \in X \) with \( x \neq y \),
(G3) \( G(x, x, y) \leq G(x, y, z) \) for all \( x, y, z \in X \) with \( y \neq z \),
(G4) \( G(x, y, z) = G(x, y, z) = G(y, z, x) \) =... (symmetry in all three variables ),
(G5) \( G(x, y, z) \leq G(x, a, a) + G(a, y, z) \) for all \( x, y, z, a \in X \)(rectangle inequality).

Then the function G is called a generalized metric or, more specially, a G-metric on X, and the pair \((X, G)\) is called a G-metric space.

The idea of complex metric space was initiated by Azam et.al.[1] to exploit the idea of complex valued normed spaces and complex valued Hilbert spaces.

Definition 1.2. Let \( \mathbb{C} \) be the set of complex numbers and \( z_1, z_2 \in \mathbb{C} \). Define a partial order \( \leq \) on \( \mathbb{C} \) as follows:

\( z_1 \leq z_2 \) if and only if \( \text{Re} (z_1) \leq \text{Re} (z_2) \) and \( \text{Im} (z_1) \leq \text{Im} (z_2) \)

That is \( z_1 \leq z_2 \) if one of the following holds

(C1): \( \text{Re} (z_1) = \text{Re} (z_2) \) and \( \text{Im} (z_1) = \text{Im} (z_2) \)

(C2): \( \text{Re} (z_1) < \text{Re} (z_2) \) and \( \text{Im} (z_1) = \text{Im} (z_2) \)

(C3): \( \text{Re} (z_1) = \text{Re} (z_2) \) and \( \text{Im} (z_1) < \text{Im} (z_2) \)

(C4): \( \text{Re} (z_1) < \text{Re} (z_2) \) and \( \text{Im} (z_1) < \text{Im} (z_2) \)

In particular, we will write \( z_1 \preceq z_2 \) if \( z_1 \neq z_2 \) and one of (C2), (C3) and (C4) is satisfied and we will write \( z_1 \prec z_2 \) if only (C4) is satisfied.

Remark 1.3. We obtained that the following statements hold:

(i) \( a, b \in \mathbb{R} \) and \( a \leq b \Rightarrow az \leq bz \) for all \( z \in \mathbb{C} \)
(ii) \( 0 \preceq z_1 \preceq z_2 \Rightarrow |z_1| < |z_2| \)
(iii) \( z_1 \preceq z_2 \) and \( z_2 < z_3 \Rightarrow z_1 < z_3 \).

Now we introduce the notion of complex valued G-metric space akin to the notion of complex valued metric spaces [1] as follows:

Definition 1.4. Let X be a non-empty set. Let G: \( X \times X \times X \rightarrow \mathbb{C} \) be a function satisfying the following properties:

(CG1) \( G(x, y, z) = 0 \) if \( x = y = z \),
(CG2) \( 0 < G(x, y, z) \) for all \( x, y \in X \) with \( x \neq y \),
(CG3) \( G(x, x, y) \leq G(x, y, z) \) for all \( x, y, z \in X \) with \( y \neq z \),
(CG4) \( G(x, y, z) = G(x, y, z) = G(y, z, x) \) =... (symmetry in all three variables ),
(CG5) \( G(x, y, z) \leq G(x, a, a) + G(a, y, z) \) for all \( x, y, z, a \in X \)(rectangle inequality).

Then the function G is called a complex valued general metric or, more specially, a complex valued G-metric on X, and the pair \((X, G)\) is called a complex valued G-metric space.

2. THE COMPLEX VALUED G-METRIC TOPOLOGY
A point \( x \in X \) is called interior point of a set \( A \subseteq X \), whenever there exists \( 0 < r \in \mathbb{C} \) such that

\[ B(x, r) = \{ y \in X : G(x, y, y) < r \} \subseteq A. \]

A point \( x \in X \) is called limit point of a set \( A \) whenever there exists \( 0 < r \in \mathbb{C} \),

\[ B(x, r) \cap (A/X) \neq \emptyset. \]
A is called open whenever each element of A is an interior point of A. A subset B of X is called closed whenever each limit point of B belongs to B.

**Proposition 2.1.** Let \((X, G)\) be complex valued G-metric space, then for any \(x_0 \in X\) and \(r > 0\), we have

1. \(G(x_0, x, y) < r\), then \(x, y \in B_G(x_0, r)\)
2. \(y \in B_G(x_0, r)\), then there exists a \(\delta > 0\) such that \(B_G(y, \delta) \subseteq B_G(x_0, r)\).

**Proposition 2.2.** Let \((X, G)\) be complex valued G-metric space, then for any \(x_0 \in X\) and \(r > 0\), we have,

\[
B_G \left( x_0, \frac{r}{2} \right) \subseteq B_{d_G}(x_0, r) \subseteq B_G(x_0, r).
\]

Where \(d_G(x, y) = G(x, y, y) + G(x, x, y)\).

3. CONVERGENCE, CONTINUITY AND COMPLETENESS IN COMPLEX VALUED G-METRIC SPACES

**Definition 3.1.** Let \((X, G)\) be a complex valued G-metric space, let \(\{x_n\}\) be a sequence of points of X, we say that \(\{x_n\}\) is complex valued G-convergent to \(x\) if for any \(\epsilon > 0\), there exists \(k \in \mathbb{N}\) such that \(G(x, x_n, x_k) < \epsilon\), for all \(n, m \geq k\). We refer to \(x\) as the limit of the sequence \(\{x_n\}\) and we write \(x_n \rightarrow x\).

**Proposition 3.1.** Let \((X, G)\) be complex valued G-metric space, then for a sequence \(\{x_n\}\subseteq X\) and point \(x \in X\), the following are equivalent:

1. \(\{x_n\}\) is complex valued G-convergent to \(x\)
2. \(G(x, x_n, x) \rightarrow 0\) as \(n \rightarrow \infty\)
3. \(G(x_n, x) \rightarrow 0\) as \(n \rightarrow \infty\)
4. \(G(x_n, x_m) \rightarrow 0\) as \(n, m \rightarrow \infty\)

**Definition 3.2.** Let \((X, G)\) and \((X', G')\) be two complex valued G-metric spaces. Then a function \(f: X \rightarrow X'\) is complex valued G-continuous at point \(x \in X\) if for any \(\epsilon > 0\), there exists \(k \in \mathbb{N}\) such that \(G(x, x_n, x_k) < \epsilon\) for all \(n, m \geq k\).

Since complex valued G-metric topologies are metric topologies we have:

**Proposition 3.2.** Let \((X, G)\) and \((X', G')\) be two complex valued G-metric spaces. Then a function \(f: X \rightarrow X'\) is complex valued G-continuous at a point \(x \in X\) if and only if \(f(x_n)\) is complex valued G-continuous at \(x\) whenever \(\{x_n\}\) is complex valued G-convergent to \(x\).

**Proposition 3.3.** Let \((X, G)\) be a complex valued G-metric spaces, then the function \(G(x, y, z)\) is jointly continuous in all three of its variables.

**Proof.** Suppose \(\{x_k\}, \{x_m\}\), and \(\{x_n\}\), are complex valued G-convergent to \(x, y\) and \(z\) respectively. Then, by (CG5) we have,

\[
G(x, y, z) \leq G(y, y_m, y_m) + G(y_m, x, z)
\]

and

\[
G(z, x, y_m) \leq G(x, x_k, x_k) + G(x_k, y_m, x)\]

so,

\[
G(x, y, z) = G(x_k, y_m, y_m) + G(x_k, x_k) + G(z, x_n, x_n)\]

Similarly,

\[
G(x_k, y_m, x_n) - G(x, y, z) \leq G(x_k, x, x) + G(y_m, y, y) + G(z, z_n, z_n).
\]

But then combining these using (3) of proposition 4.1 we have,

\[
G(x_k, y_m, x_n) - G(x, y, z) \leq 2G(x_k, x_k) + G(y, y_m, y_m) + G(z, z_n, z_n),
\]

so,

\[
G(x_k, y_m, x_n) - G(x, y, z) \rightarrow 0, \text{ as } k, m, n, \rightarrow \infty\] and the result follows by proposition 3.2.

**Definition 3.2.** Let \((X, G)\) be a complex valued G-metric space, a sequence \(\{x_n\}\) is complex valued G-Cauchy if given \(\epsilon > 0\), there exists \(k \in \mathbb{N}\) such that \(G(x, x_n, x_k) < \epsilon\) for all \(n, m \geq k\).

**Definition 3.3.** A complex valued G-metric space \((X, G)\) is said to be complex valued G-complete if every complex valued G-Cauchy sequence is complex valued G-convergent in \((X, G)\).

**Proposition 3.4.** Let \((X, G)\) be a complex valued G-metric space. Then the following are equivalent:

1. The sequence \(\{x_n\}\) is a complex valued G-Cauchy in \(X\).
2. For every \(\epsilon > 0\), there exists \(k \in \mathbb{N}\) such that \(G(x_n, x_m, x_k) < \epsilon\) for all \(n, m \geq k\).
3. \(\{x_n\}\) is a Cauchy sequence in the complex valued metric space \((X, d_G)\).

**Proposition 3.5.** Let \((X, G)\) be a complex valued G-metric space and \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) is complex valued G-convergent to \(x\) if and only if \(G(x, x_m, x_n) \rightarrow 0\) as \(n, m \rightarrow \infty\).

**Proof.** Suppose \(\{x_n\}\) is complex valued G-convergent to \(x\). For a given real number \(\epsilon > 0\), let

\[
x = \frac{\epsilon}{\sqrt{3}} + \frac{\epsilon}{\sqrt{2}}.\]

Then \(0 < r \in \mathbb{C}\) and there is a natural number \(k\), such that\(G(x, x_m, x_n) < \epsilon\) for all \(n, m \geq k\).
Therefore,
\[ |G(x, x_n, x_m)| < |c| = \epsilon \text{ for all } n, m \geq k. \]
It follows that \( |G(x, x_n, x_m)| \to 0 \text{ as } n, m \to \infty. \)

Conversely, suppose that \( |G(x, x_n, x_m)| \to 0 \text{ as } n, m \to \infty. \)
Then given \( r \in \mathbb{C} \) with \( 0 < c, \) there exists a real number \( \delta > 0, \) such that for \( z \in \mathbb{C} \)
\[ |z| < \delta \implies z < c. \]
For this \( \delta, \) there is a natural number \( k \) such that
\[ |G(x, x_n, x_m)| < \delta \text{ for all } n, m \geq k. \]
This means that \( G(x, x_n, x_m) < \epsilon \text{ for all } n, m \geq k. \) Hence \( \{x_n\} \)
is complex valued \( G- \)convergent to \( x. \)

**Proposition 3.6.** Let \( (X, G) \) be a complex valued \( G- \)metric space and \( \{x_n\} \) be a sequence in \( X. \) Then \( \{x_n\} \) is complex valued \( G- \)Cauchy sequence if and only if
\[ |G(x, x_n, x_m)| \leq \epsilon \text{ for all } n, m \geq k. \]

**Proof.** Suppose that \( \{x_n\} \) is complex valued \( G- \)Cauchy sequence. For a given real number \( \epsilon > 0, \) let
\[ c = \frac{\epsilon}{\sqrt{2}} + \frac{\epsilon}{\sqrt{2}}. \]
Then \( 0 < r \in \mathbb{C} \) and there is a natural number \( k, \) such that
\[ |G(x, x_n, x_m)| < \epsilon \text{ for all } n, m \geq k. \]

Therefore,
\[ |G(x, x_n, x_m)| < |c| = \epsilon \text{ for all } n, m \geq k. \]
It follows that \( |G(x, x_n, x_m)| \to 0 \text{ as } n, m \to \infty. \)

Conversely, suppose that \( |G(x, x_n, x_m)| \to 0 \text{ as } n, m \to \infty. \)
Then given \( c \in \mathbb{C} \) with \( 0 < c, \) there exists a real number \( \delta > 0, \) such that for \( z \in \mathbb{C} \)
\[ |z| < \delta \implies z < c. \]
For this \( \delta, \) there is a natural number \( k \) such that
\[ |G(x, x_n, x_m)| < \delta \text{ for all } n, m \geq k. \]
This means that \( G(x, x_n, x_m) < \epsilon \text{ for all } n, m \geq k. \) Hence \( \{x_n\} \) is complex valued \( G- \)Cauchy sequence.

### 4. PROPERTIES OF COMPLEX VALUED \( G- \)METRIC SPACES

**Proposition 4.1.** Let \( (X, G) \) be a complex valued \( G- \)metric space. Then for any \( x, y, z, a \) in \( X \) it follows that:

(i) If \( G(x, y, z) = 0 \) if \( x = y = z \)
(ii) \( G(x, y, z) \leq G(x, x, y) + G(x, x, z) \)
(iii) \( G(x, y, y) \leq 2G(x, y, x) \)
(iv) \( G(x, y, z) \leq G(x, a, z) + G(a, y, z) \)
(v) \( G(x, y, z) \leq 2/3 (G(x, y, a) + G(x, a, z) + G(a, y, z)) \)
(vi) \( G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a)). \)

**Proposition 4.2.** Let \( (X, G) \) be a complex valued \( G- \)metric space. Then the following are equivalent:

(i) \( (X, G) \) is symmetric.
(ii) \( G(x, y, y) \leq k G(Tx, Ty, Tz) \text{ for all } x, y, z \in X, \) where \( 0 \leq k < 1. \)
(iii) \( G(x, y, z) \leq G(x, y, a) + G(z, y, b) \text{ for all } x, y, a, b \in X. \)

In 1998, Jungck [7] introduced the concept of weakly compatibility as follows:

**Definition 4.3** Two self mappings \( S \) and \( T \) are said to be weakly compatible if they commute at their coincidence points.

### 5. MAIN RESULT

Now we prove our main result for a pair of self mappings:

**Theorem 5.1.** Let \( (X, G) \) be a complete complex valued \( G- \)metric space. Let \( S, T: X \to X \) be self mappings satisfying the following conditions:

(2.1) \( S(X) \subseteq T(X), \)
(2.2) any one of the subspace \( S(X) \) or \( T(X) \) is complete,
(2.3) \( G(Sx, Sy, Sz) \leq k G(Tx, Ty, Tz) \text{ for all } x, y, z \in X, \) where \( 0 \leq k < 1. \)
(2.4) \( S \) and \( T \) are weakly compatible self maps.

Then \( S \) and \( T \) have a unique common fixed point in \( X. \)

**Proof.** Let \( x_0 \in X \) be an arbitrary point in \( X. \) By (2.1), one can choose a point \( x_1 \) in \( X \) such that \( Sx_0 = Tx_1. \)
In general choose \( x_{n+1} \) such that
\[ y_n = Sx_n = Tx_{n+1}. \]
Now, we prove \( \{y_n\} \) is a complex valued \( G- \)Cauchy sequence in \( X. \)

Putting \( x = x_n, y = x_{n+1}, z = x_{n+1} \) in (2.1), we have
\[ G(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq k G(Tx_n, Tx_{n+1}, Tx_{n+1}) = k G(Sx_{n-1}, Sx_n, Sx_n). \]
Continuing in the same way, we have
\[ G(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq k^n G(Sx_0, Sx_1, Sx_1). \]
This implies that \( G(y_n, y_{n+1}, y_{n+1}) \leq k^n G(y_0, y_1, y_1) \)
Then, for all \( n, m \in N, n < m, \) we have by \( \text{CG5} \)
\[ G(y_n, y_m, y_m) \leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + G(y_{n+2}, y_{n+3}, y_{n+3}) + \ldots + G(y_{m-1}, y_m, y_m) \leq (k^n + \ldots + k + 1)G(y_0, y_1, y_1). \]
Therefore,
\[ |G(y_n, y_m, y_m) - G(y_n, y_m, y_m)| \leq \left( \frac{k^n}{1-k} \right) |G(y_0, y_1, y_1)| \]

Since \( k \in [0, 1] \), if we taking limit as \( n, m \to \infty \), then
\[ \frac{k^n}{1-k} \to 0 \]

i.e., \( G(y_n, y_1, y_1) \to 0 \)

For \( n, m, l \in \mathbb{N} \) (CG5) implies that
\[ G(y_n, y_m, y_m) \leq G(y_n, y_m, y_m) + G(y_1, y_m, y_m) \]

Taking limit as \( n, m, l \to \infty \), we get \( G(y_n, y_m, y_m) \to 0 \) i.e.,
\[ G(y_n, y_m, y_m) \to 0 \]

So \( \{y_n\} \) is complex valued G-Cauchy sequence. Since either \( S(X) \) or \( T(X) \) is complete. Without loss of generality, we assume that \( T(X) \) is complete subspace of \( X \), then the subsequence of \( \{y_n\} \) must get a limit in \( T(X) \) (say) \( z \). Then \( Tu = z \) for some \( u \in X \), as \( \{y_n\} \) is a complex valued G-Cauchy sequence containing a convergent subsequence, therefore the sequence \( \{y_n\} \) also convergent implying thereby the convergence of subsequence of the convergent sequence. Next we show that \( Su = z \). On setting \( x = u, y = x_n \) and \( z = x_n \) in (2.3), we have
\[ G(Su, x_n, x_n) \leq kG(Tu, Tu, Tu) \]

Taking limit as \( n \to \infty \), we have
\[ G(Su, z, z) \leq kG(Tu, z, z) \]

Therefore, \( Su = Tu = z \). i.e., \( u \) is coincidence point of \( S \) and \( T \). Since \( S \) and \( T \) are weakly compatible, it follows that \( Tu = TSu \) i.e., \( Sz = Tz \).

We now show that \( Sz = z \). Suppose that \( S(z) \neq z \), therefore \( 0 < G(Sz, z, z) \) implies that \( G(Sz, z, z) > 0 \).

Putting \( x = z, y = u, z = u \) in (2.3), we have
\[ G(Sz, Su, Su) \leq kG(Tz, Tu, Tu) = kG(Sz, z, z) \]

e.i., \( G(Sz, z, z) \leq kG(Sz, z, z) < G(Sz, z, z) \) which is a contradiction, therefore
\[ Sz = z \). Thus \( Sz = Tz = z \) i.e., \( z \) is a common fixed point of \( S \) and \( T \).

**Uniquexe**: To prove uniqueness, suppose that \( w \neq z \) be another common fixed point of \( S \) and \( T \). Then \( 0 < G(z, w, w) \) implies that \( G(z, w, w) > 0 \).

Putting \( x = z, y = u, z = u \) in (2.3), we have
\[ G(z, w, w) = G(Sz, Sw, Sw) \leq kG(Tz, Tw, Tw) = kG(z, w, w) \]

e.i., \( G(z, w, w) \leq kG(z, w, w) < G(z, z, z) \), which is a contradiction, therefore
\[ z = w \). Thus \( Sz = Tz = z \) i.e., \( z \) is a unique common fixed point of \( S \) and \( T \).

**Example 5.1.** Let \( X = [-1, 1] \) and let \( G: X \times X \times X \to \mathbb{C} \) be complex valued G-metric space defined as follows:
\[ G(x, y, z) = |x - y| + |y - z| + |z - x| \]
for all \( x, y, z \in X \). Then \( (X, G) \) is complex valued G-metric space.

Define \( S, T: X \to X \) as \( Sx = \frac{x}{2} \) and \( Tx = \frac{x}{6} \).

Here we note that, (2.1) \( S(X) \subseteq T(X) \), (2.2) Both \( S(X) \) and \( T(X) \) are complete.

(2.3) \( G(Sx, Sy, Sz) \leq kG(Tx, Ty, Tz) \) holds for all \( x, y, z \in X \), \( 1/3 \leq k < 1 \), (2.4) \( S \) and \( T \) are weakly compatible because \( S \) and \( T \) commute at their coincidence point i.e., at \( x=0 \) and \( x=0 \) is the unique common fixed point of \( S \) and \( T \).

**6. REFERENCES**


