Approximate Solution of Volterra-Fredholm Integral Equation with Hilbert Kernel

A.S. Ismail
Department of Mathematics, Faculty of Science, Zagazig University, Egypt
Temporary address: Mathematics Department, University College in Al-Qunfudah, Umm Al-Qura University, KSA

ABSTRACT
In this work, we use numerical technique to reduce the Volterra-Fredholm integral equation to a linear system of Fredholm integral equations of the second kind and we apply the product Nystrom method to solve this system of integral equations to get the approximate solution of Volterra-Fredholm integral equation. The results are compared with the exact solution of the integral equation.

Keywords:

1. INTRODUCTION
Consider the following Volterra-Fredholm integral equation with Hilbert kernel of the form
\[
\phi(x, t) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \cot \frac{y-x}{2} \phi(y, \tau) \, dy \, d\tau = f(x, t),
\]
\[
(x, t) \in [-\pi, \pi] \times [0, T],
\]
where \( \phi \) is an unknown function which is sought in \( L_2 = L_2(-\pi, \pi; 0, T) \) with the usual norm under the condition \( \phi(\pm \pi, t) = 0 \), and \( f(x, t) \in L_2 \) is given function.

Numerical methods for Volterra-Fredholm integral equations have been studied in [3,6-11]. For the linear case, some projection methods are given in [7,8,9]. Kauthen [9] studied continuous time collocation, time discretization collocation methods, and he analyzed their global discrete convergence properties, local and global superconvergence properties. Brunner [3] studied the numerical solution of general integral equations of this type by continuous-time and discrete-time spline collocation methods. The results of Kauthen [9] have been extended to nonlinear Volterra-Fredholm integral equations by Brunner [3]. The Adomian decomposition method for mixed nonlinear Volterra-Fredholm integral equations was presented by K. Maleknejad [10].

2. THE SYSTEM OF INTEGRAL EQUATIONS
We use numerical technique [4] to reduce the V-FIE (1) to a linear system of Fredholm integral equations of the second kind. At \( t = 0 \), then Eq.(1) is reduced to
\[
\phi(x, 0) = f(x, 0). 
\]

For \( t \neq 0 \), we consider the numerical solution of Eq.(1). An obvious numerical procedures is to approximate the integral of Volterra term in Eq.(1) via a quadrature rule which integrates over the variable \( \tau \) for a fixed value of \( t \). Thus setting \( t = t_r \equiv rt \), where \( t = \frac{1}{M} \) is the fixed steplength, we would approximate, in an obvious notation, the integral of Volterra term, in Eq.(1), by
\[
\int_{0}^{t_r} \int_{-\pi}^{\pi} \cot \frac{y-x}{2} \phi(y, \tau) \, dy \, d\tau \approx \int_{r}^{r+1} \int_{-\pi}^{\pi} u_{rj} \cot \frac{y-x}{2} \phi(y, \tau) \, dy \]
\[
\cdot \cdot \cdot + \int_{r-M}^{r-1} \int_{-\pi}^{\pi} u_{rj} \cot \frac{y-x}{2} \phi(y, \tau) \, dy \]
at \( r = 1 \)

\[
\phi^{(1)}(x) = \frac{\ell}{4\pi} \int_{-\pi}^{\pi} \cot \frac{y - x}{2} \phi^{(1)}(y) dy = f^{(1)}(x) + \frac{\ell}{4\pi} H(x) \tag{8}
\]

where

\[
H(x) = \int_{-\pi}^{\pi} \cot \frac{y - x}{2} f^{(0)}(y) dy. \tag{9}
\]

At \( r \geq 2 \) we obtain

\[
\phi^{(r)}(x) = \frac{\ell}{4\pi} \int_{-\pi}^{\pi} \cot \frac{y - x}{2} \phi^{(r)}(y) dy = g^{(r)}(x), \tag{10}
\]

where

\[
g^{(r)}(x) = f^{(r)}(x) + \sum_{j=1}^{r-1} (-1)^{j/2} \left[ f^{(r-j)}(x) - \phi^{(r-j)}(x) \right] + \left(-1\right)^{r+1} \frac{\ell}{4\pi} H(x), \quad r = 2, 3, \ldots \tag{11}
\]

Equations (8) and (10) represents a system of Fredholm integral equations of the second kind that can be solved by many numerical techniques. In the following section we shall apply the product Nyström method [1,2,4] to obtain the approximate solutions of Eqs. (8) and (10) which leads to the required approximate solution of the Volterra-Fredholm integral equation (1).

3. THE PRODUCT NYSTROM METHOD

According to the product Nyström method [1,2,4], we approximate the integral term in Eq.s (8) and (10) when \( x = x_i \) by

\[
\int_{-\pi}^{\pi} \cot \frac{y - x_i}{2} \phi^{(r)}(y) dy \approx \sum_{j=0}^{N} w_{ij} \phi^{(r)}(y_j), \tag{12}
\]

where \( w_{ij} \) are the weights. Also, we approximate the integral term in (12) by a product integration form of Simpson’s rule, we may write

\[
\int_{-\pi}^{\pi} \cot \frac{y - x_i}{2} \phi^{(r)}(y) dy = \frac{\pi}{2} \sum_{j=0}^{N/2} \int_{y_{2j}}^{y_{2j+2}} \cot \frac{y - x_i}{2} \phi^{(r)}(y) dy, \tag{13}
\]

where \( x_i = y_i = -\pi + ih, \quad i = 0, 1, \ldots, N \) with \( h = \frac{2\pi}{N} \) and \( N \) even. Now if we approximate the nonsingular part of the integrand over each interval \([ y_{2j}, y_{2j+2} ]\) by the second degree Lagrange interpolation polynomial which interpolates it at the points \( y_{2j}, y_{2j+1}, y_{2j+2} \), we find

\[
\int_{-\pi}^{\pi} \cot \frac{y - y_i}{2} \phi^{(r)}(y) dy \approx \sum_{j=0}^{N/2} \int_{y_{2j}}^{y_{2j+2}} \left\{ \frac{(y_{2j+1} - y)(y_{2j+2} - y)}{2h^2} \phi^{(r)}(y_{2j}) + \frac{(y - y_{2j})(y_{2j+2} - y)}{h^2} \phi^{(r)}(y_{2j+1}) + \frac{(y - y_{2j})(y - y_{2j+1})}{2h^2} \phi^{(r)}(y_{2j+2}) \right\} dy
\]

\[
= \sum_{j=0}^{N} w_{ij} \phi^{(r)}(y_j). \tag{14}
\]

where

\[
w_{i,0} = \frac{1}{2h^2} \int_{y_{2j}}^{y_{2j+2}} \cot \frac{y - y_i}{2} (y_1 - y)(y_2 - y) dy,
\]

\[
w_{i,2j+1} = \frac{1}{h^2} \int_{y_{2j}}^{y_{2j+2}} \cot \frac{y - y_i}{2} (y - y_2)(y_{2j+2} - y) dy,
\]

\[
w_{i,2j} = \frac{1}{2h^2} \int_{y_{2j}}^{y_{2j+2}} \cot \frac{y - y_i}{2} (y - y_{2j})(y_{2j+2} - y) dy
\]

\[
+ \frac{1}{h^2} \int_{y_{2j}}^{y_{2j+2}} \cot \frac{y - y_i}{2} (y_{2j+1} - y)(y_{2j+2} - y) dy,
\]

\[
w_{i,N} = \frac{1}{2h^2} \int_{y_{2j}}^{y_{2j+2}} \cot \frac{y - y_i}{2} (y - y_{N-2})(y - y_{N-1}) dy.
\]

If we define

\[
\alpha_j(y_i) = \frac{1}{2h^2} \int_{y_{2j}}^{y_{2j+2}} \cot \frac{y - y_i}{2} (y - y_{2j-2})(y - y_{2j-1}) dy,
\]

\[
\beta_j(y_i) = \frac{1}{2h^2} \int_{y_{2j}}^{y_{2j+2}} \cot \frac{y - y_i}{2} (y - y_{2j-1})(y_{2j} - y) dy,
\]

\[
\gamma_j(y_i) = \frac{1}{2h^2} \int_{y_{2j}}^{y_{2j+2}} \cot \frac{y - y_i}{2} (y - y_{2j-2})(y_{2j} - y) dy.
\]

it follows that

\[
w_{i,0} = \beta_1(y_i),
\]

\[
w_{i,2j+1} = 2\gamma_1(y_i),
\]

\[
w_{i,2j} = \alpha_j(y_i) + \beta_{j+1}(y_i),
\]

\[
w_{i,N} = \alpha_N(y_i). \tag{16}
\]

We introduce the change of variable \( y = y_{2j-2} + \zeta h, \quad 0 \leq \zeta \leq 2, \) thus the system (15) becomes

\[
\alpha_j(y_i) = \frac{h}{2} \int_{0}^{2} (\zeta - 1) \cot \frac{h(\zeta - i + 2 j - 2)}{2} d\zeta,
\]

\[
\beta_j(y_i) = \frac{h}{2} \int_{0}^{2} (1 - \zeta)(2 - \zeta) \cot \frac{h(\zeta - i + 2 j - 2)}{2} d\zeta,
\]

\[
\gamma_j(y_i) = \frac{h}{2} \int_{0}^{2} (2 - \zeta) \cot \frac{h(\zeta - i + 2 j - 2)}{2} d\zeta. \tag{17}
\]

After evaluate the previous integrals we can obtain

\[
w_{i,2j+1} = 2z(2 - z) \ln \left| \sin \frac{h(2 - z)}{2} \right|
\]

\[-2z(2 - z) \ln \left| \sin \frac{h\zeta}{2} \right|
\]

\[+ \sum_{s=0}^{\infty} \frac{(-1)^s h^{2s} B_{2s}}{(2s)!} \frac{(2 - z)^{s+2s}(1 - z) + z^{1 + 2s}(1 - z)}{1 + 2s} \]

\[- \frac{(2 - z)^{2s+2} z^{s+2s}}{2(2 + 2s)}, \tag{18}
\]

and

\[
w_{i,2j} = 6(z - 2) \ln \left| \sin \frac{h(2 - z)}{2} \right|
\]
\[+(z-3)(z-4)\ln\left|\frac{\sin\left(\frac{h(4-z)}{2}\right)}{2}\right| - z(z-1)\ln\left|\frac{hz}{2}\right|
\]
\[+ \sum_{s=0}^{\infty} \frac{(-1)^s h^{2s} B_{2s}}{(2s)!} \times \]
\[\left((6(2-z)^{1+2s} + z^{1+2s}(2z-1) + (4-z)^{1+2s}(2z-7)\right)
\]
\[\frac{1}{1+2s} - \frac{2s^{2s} - 2(4-z)^{2+2s}}{2(2+2s)}\right), \tag{19}\]

where \(B_{2s}\) are Bernoulli numbers \([5]\), \(z = i - 2j + 2\) and \(1 \leq i, j \leq N - 1\). The condition \(\phi(\pm\pi, t) = 0\), avoids the use of the two points \(x = \pm\pi\) and avoids the calculation of \(w_{i,0}\) and \(w_{i,N}\). Therefore, the solutions of the integral equations (8) and (10), at every \(r\), can be reduced to the solutions of the following system of linear algebraic equations of the form

\[\phi^{(r)}(x_i) - \frac{\ell}{4\pi} \sum_{j=1}^{N-1} w_{ij} \phi^{(r)}(y_j) = g^{(r)}(x_i), \quad i = 0, 1, \ldots, N,\]

or

\[\phi^{(r)} - \frac{\ell}{4\pi} W \phi^{(r)} = G^{(r)},\]

which has the solution

\[\phi^{(r)} = [I - \frac{\ell}{4\pi} W]^{-1} G^{(r)},\]

where \(I\) is the identity matrix and \(\det(I - \frac{\ell}{4\pi} W) \neq 0\).

4. NUMERICAL RESULTS

EXAMPLE 1. Consider the Volterra-Fredholm integral equation

\[\phi(x, t) - \frac{1}{2\pi} \int_0^t \int_{-\pi}^{\pi} \cot\left(\frac{y-x}{2}\right) \phi(y, \tau) \, dy \, d\tau = \sin x. \tag{21}\]

One can see that \(\phi(x, t) = \sin(x + t)\) is the exact solution of Eq (21). To achieve the validity, the accuracy and support our theoretical discussion of the proposed method, we notice that approximate solution is nothing but the exact solution as shown in Figure 1. The problem (21) solved numerically with \(M = 20\) and \(T = 0.5\) for different values of \(N\) to see effect and this is shown in Figure 2 for different values of parameter \(r\) in the volterra integral, moreover the parameter \(t\) is fixed at \(t = t_i = i\ell, i = 2, 9, 16\). Clearly as the number of intervals \((N)\) increases, the accuracy of our solution to the integral equation also increases, such a result that should not surprise us.

5. REFERENCES


