Fixed Points of Non Self Maps

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ABSTRACT
The purpose of this paper is to present some fixed point theorems for non self maps in $d_p$-complete topological spaces which extend the results of Linda Marie Saliga.

Keywords
$d_p$- complete topological spaces, $d$-complete topological spaces and non self maps.

1. INTRODUCTION
Troy L. Hicks [5] has introduced $d$-complete topological spaces, attributing the basic ideas of these spaces to Kasahara ([8], [9]) and Iseki [7] as follows:

1.1 Definition: A topological space $(X, t)$ is said to be $d$-complete if there is a mapping $d : X \times X \to [0, \infty)$ such that
(i) $d(x, y) = 0 \iff x = y$ and
(ii) $\{x_n\}$ is a sequence in $X$ such that $\sum_{n=1}^{\infty} d(x_n, x_{n+1})$ is convergent implies that $\{x_n\}$ converges to some point in $(X, t)$. In this paper we introduce $d_p$-complete topological spaces as a generalization of $d$-complete topological spaces for any integer $p \geq 2$. For a non-empty set $X$, let $X^p$ be its $p$-fold cartesian product.

1.2 Definition: A topological space $(X, t)$ is said to be $d_p$-complete if there is a mapping $d_p : X^p \to [0, \infty)$ such that
(i) $d_p(x_1, x_2, \ldots, x_p) = 0 \iff x_1 = x_2 = \ldots = x_p$ and
(ii) $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} d_p(x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+p-1}) = 0$ implies that $\{x_n\}$ converges to some point in $(X, t)$. A $d_p$-complete topological space is denoted by $(X, t, d_p)$.

1.3 Remark: The function $d$ in the Definition 1.1 and the function $d_2$ (the case $p = 2$) in Definition 1.2 are both defined on $X \times X$ and satisfy condition (i) of the definitions which are identical. Since the convergence of an infinite series $\sum_{n=1}^{\infty} a_n$ of real numbers implies $\lim_{n \to \infty} a_n = 0$, but not conversely; it follows that every $d$-complete topological space is $d_2$-complete, but not conversely. Therefore the class of $d_2$-complete topological spaces is wider than the class of $d$-complete spaces and hence a separate study of fixed point theorems of self-maps on $d_2$-complete topological spaces is meaningful.

The purpose of this paper is to establish certain fixed point theorems of non self-maps of $d_p$-complete topological spaces for $p \geq 2$.

2. PRELIMINARIES
Let $X$ be a non-empty set. A mapping $d_p : X^p \to [0, \infty)$ is called a $p$-non-negative on $X$ provided $d_p(x_1, x_2, \ldots, x_p) = 0 \iff x_1 = x_2 = \ldots = x_p$.

2.1 Definition: Suppose $(X, t)$ is a topological space and $d_p$ is a $p$-non-negative on $X$. A sequence $\{x_n\}$ in $X$ is said to be a $d_p$- Cauchy sequence if $d_p(x_n, x_{n+1}, \ldots, x_{n+p-1}) \to 0$ as $n \to \infty$.

In view of Definition 2.1, a topological space $(X, t)$ is $d_p$-complete if there is a $p$-non-negative $d_p$ on $X$ such that every $d_p$- Cauchy sequence in $X$ converges to some point in $(X, t)$.

If $T$ is a self map of a non-empty set $X$ and $x \in X$, then the orbit of $x$, $O_T(x)$ is given by $O_T(x) = \{x, Tx, T^2x, \ldots, \}$. If $T$ is a self map of a topological space $X$, then a mapping $G : X \to [0, \infty)$ is said to be $T$-orbitally lower semi-continuous (resp. $T$-orbitally continuous) at $x^* \in X$ if $G(x^*)$ is a sequence in $O_T(x)$ for some $x \in X$ with $x_n \to x^*$ as $n \to \infty$ then $G(x^*) \leq \liminf_{n \to \infty} G(x_n)$ (resp. $G(x^*) = \lim_{n \to \infty} G(x_n)$). A self map $T$ of topological space $X$ is said to be $w$-continuous at $x \in X$ if $x_n \to x$ as $n \to \infty$ implies $Tx_n \to Tx$ as $n \to \infty$.

If $d_p$ is a $p$-non-negative on a non-empty set $X$, and $T : X \to X$ then we write, for simplicity of notation, that

(2.2) $G_p(x) := d_p(x, Tx, T^2x, \ldots, T^{p-1}x) \text{ for } x \in X$

Clearly we have

(2.3) $G_p(x) = 0 \text{ if and only if } x \text{ is a fixed point of } T$.

3. MAIN RESULTS
3.1 Theorem: Suppose $(X, t, d_p)$ is a $d_p$-complete Hausdorff topological space, $C$ is a closed subset of $X$ and $T : C \to X$ is an open mapping with $C \subset T(C)$. Suppose $d_p(x_1, x_2, \ldots, x_p) \leq k(d_p(Tx_1, Tx_2, \ldots, Tx_p))$ for all
Note that \( \alpha \), \( \beta \) and \( \gamma \) are such that \( T : X \to X \) and \( T (a) = a \) for all \( a \in X \).

Thus, \( T \) is one continuous function and hence \( T \) is \( t \)-semi-continuous function.

Now first suppose that \( T \) has a fixed point \( z \in C \).

Then \( Tz = z \) so that \( k^4(Tz,x,y,…,y) \leq k(d(Tz,Tx,…,Tx)) \) which gives that \( T \) is \( t \)-semi-continuous function.

Hence, by induction, we get \( (3.5) \).

Therefore, \( k^4(d(T^{p+4}x_0, T^{p+3}x_0,…, Tx_0, x_0)) \leq \beta \), since \( 0 < t < 1 \).

Hence, \( T \) is \( t \)-semi-continuous function.

3.6 Theorem: Let \( (X,d) \) be a complete Hausdorff \( t \) topology space, \( C \) be a closed subset of \( X \) and \( T : X \to X \) with \( C \subset T \). Suppose there exists a function \( k : [0,\infty) \to [0,\infty) \) such that \( k(d(Tx, Tx,…, Tx)) \geq d(x_1,x_2,…,x_n) \) for all \( x \in X \).

Then \( T \) is \( t \)-semi-continuous function.

3.7 Remark: Note that the result proved by Linda Marie Saliga \([10]\), Theorem 2, pp.103,104) is a particular case of Theorem 3.6.

3.8 Theorem: Let \( C \) be a compact subset of a Hausdorff \( t \) topology space \( (X,d) \) and \( G_{\beta}(x) \) be a \( \beta \)-non-negative on \( X \).

Suppose \( T : C \to C \) and \( G_t(x) \) are both continuous, and \( G_t(x) \geq G_t(y) \) for all \( x \neq y \).

Then \( T \) has a fixed point in \( C \).

Proof: Since \( C \) is a compact subset of a Hausdorff \( t \) topology space, we get that \( C \) is closed and since \( T : C \to X \)
is continuous, so $T^1(C)$ is closed. Hence $T^1(C)$ is compact since $T^1(C) \subset C$. Also, $G_p(x)$ is continuous so it attains its minimum on $T^1(C)$, say at $z$. That is, 

(3.9) \hspace{1cm} G_p(z) \leq G_p(x) \text{ for all } x \in T^1(C) \hspace{1cm}

Now $z \in C \subset T(C)$ so there exists $y \in T^1(C)$ such that $Ty = z$. If $y \neq z$, then $G_p(z) = G_p(Ty) > G_p(y)$ which is a contradiction to (3.9).

Thus $y = z = Ty$ is a fixed point of $T$.

3.9 Remark: Note that the result proved by Linda Marie Saliga ([10], Theorem 3, pp.106) is a particular case of Theorem 3.8.

10 Theorem: Let $C$ be a compact subset of a Hausdorff topological space $(X,t)$ and $d_p$ be a p-nonnegative on $X$. Suppose $T : C \to X$ with $C \subset T(C)$, $T$ and $G_p(x)$ are both continuous, $f : [0,\infty) \to [0,\infty)$ is continuous and $f(t) > 0$ for $t \neq 0$. If we know that $G_p(Tx) \leq \lambda f(G_p(x))$ where $0 < \lambda < 1$, for all $x \in T^1(C)$ implies $T$ has a fixed point then $G_p(Tx) < f(G_p(x))$ for all $x \in T^1(C)$ such that $f(G_p(x)) \not= 0$ gives a fixed point.

Proof: Since $C$ is a compact subset of a Hausdorff space, it is closed and since $T$ is continuous, $T^1(C)$ is closed and hence is compact since $T^1(C) \subset C$. Suppose $G_p(x) > 0$ for all $x \in T^1(C)$. Then $G_p(x) > 0$ so that $f(G_p(x)) > 0$ for all $x \in T^1(C)$. Now define $p(x)$ on $T^1(C)$ by $p(x) = \frac{G_p(Tx)}{f(G_p(x))}$. Then $p$ is continuous since $T, f$ and $G_p(x)$ are continuous. Therefore $p$ attains its maximum on $T^1(C)$, say at $z$. That is, $p(x) \leq p(z)$ for all $x \in T^1(C)$. Now $p(x) \leq p(z) < 1$ so $G_p(Tx) \leq p(z)f(G_p(x))$ and $T$ must have a fixed point.

11 Remark: Note that the result proved by Linda Marie Saliga ([10], Theorem 4, pp.106) is a particular case of Theorem 3.10.

12 Theorem: Let $C$ be a compact subset of a Hausdorff topological space $(X,t)$ and $d_p$ be a p-nonnegative on $X$. Suppose $T : C \to X$ with $C \subset T(C)$, $T$ and $G_p(x)$ are both continuous, $f : [0,\infty) \to [0,\infty)$ is continuous and $f(t) > 0$ for $t \neq 0$. If we know that $G_p(Tx) \geq \lambda f(G_p(x))$ where $\lambda > 1$, for all $x \in T^1(C)$ implies $T$ has a fixed point, then $G_p(Tx) > f(G_p(x))$ for all $x \in T^1(C)$ such that $f(G_p(x)) \neq 0$ gives a fixed point.

Proof: Since $C$ is a compact subset of a Hausdorff space, we get that $C$ is closed and since $T$ is continuous, $T^1(C)$ is closed. Hence $T^1(C)$ is compact, since $T^1(C) \subset C$. Suppose $G_p(x) \not= 0$ for all $x \in T^1(C)$. Then $G_p(x) > 0$ and $f(G_p(x)) > 0$.

Now define $p(x) = \frac{G_p(Tx)}{f(G_p(x))}$.

Then $p$ is continuous, since $T, f$ and $G_p(x)$ are continuous and hence $p$ attains its minimum on $T^1(C)$, say at $z$. That is, $p(z) \leq p(x)$ for all $x \in T^1(C)$.

Now $p(x) \geq p(z) > 1$ so $G_p(Tx) \geq p(z)f(G_p(x))$ and $T$ must have a fixed point.

13 Remark: Note that the result proved by Linda Marie Saliga ([10], Theorem 5, pp.106,107) is a particular case of Theorem 3.12.

4. REFERENCES

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