Decomposability of Projective Curvature Tensor in Recurrent Finsler Space ($WR - F_n$)

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ABSTRACT
The decomposition of curvature tensor field was studied by K. Takano[1]. The decomposability of curvature tensor in Finsler manifold was studied by Pandey[2]. The purpose of the present chapter is to decompose the Projective curvature tensor in recurrent Finsler space and study the properties of conformal decomposition tensor.

Keywords
Finsler space, projective curvature tensor, recurrent Finsler space.

1. INTRODUCTION
We considered an n-dimensional Finsler space F_n in which the projective curvature tensor, projective tensor field and deviation tensor field are defined by Rund[3]

$$(1.1)\quad \begin{cases} \text{(a)}\ W^i_{jkh} = H^i_{jkh} + \frac{2\delta^i_j}{n+1} H_{[jk]} + \frac{2\dot{x}^i}{n+1}\dot{\partial}_j H_{[kh]} \\ \quad + \frac{\delta^i_k}{n^2-1}\left(nH_{jh} + H_h + \dot{x}^r\dot{\partial}_j H_{hr}\right) - \\ \quad \frac{\delta^i_h}{n^2-1}\left(nH_{jk} + H_{kj} + \dot{x}^r\dot{\partial}_j H_{kr}\right) \\ \text{(b)}\ W^i_{jk} = H^i_{jk} + \frac{\dot{x}^i}{n+1} H_{[jk]} \\ \quad +2\left\{\frac{\delta^i_{[j}}{n^2-1}\left(nH_{k]} - \dot{x}^r H_{k]r}\right)\right\} \end{cases}$$

$$(1.2)\quad W^i_j = H^i_{jk} - H\delta^i_j - \frac{1}{n+1}\left(\dot{\partial}_r H^r_j - \dot{\partial}_j H\right)\dot{x}^i,$$

respectively.

The following relations which will be used in our discussion follow from (1.1)(a) and (1.2)(b)

$$(1.3)\quad \begin{cases} \text{a)}\ W^i_{jkh}\dot{x}^j = W^i_{kh,} \\ b)\ \ W^i_{kh}\dot{x}^k = W^i_h, \\ c)\ \ W^i_h\dot{x}^h = 0. \end{cases}$$

The deviation tensor W^i_k is homogeneous of second degree in its directional arguments. The Projective tensor W^i_{jk} is skew-symmetric in its lower indices and projectively homogeneous of degree one in their directional arguments and the projective curvature tensor W^i_{jkh} is skew-symmetric in its indices k and h and is positively homogeneous of degree zero in its directional arguments.

Sinha and Singh[4] have defined that an F_n is called Projective recurrent of the first order if the Berwald's covariant derivative of the projective curvature tensor satisfies

$$(1.4)\quad W^i_{jkh(l)} = V_l W^i_{jkh},$$

Where V_l is a recurrent vector field. The space equipped with such recurrent vector field and projective curvature tensor is called recurrent Finsler space.

Transvecting (1.4) successively by $\dot{x}^j$ and $\dot{x}^k$ and therefore using (1.3)(a) and (1.3)(b), we get

$$(1.5)\quad W^i_{kh(l)} = V_l W^i_{kh},$$

$$(1.6)\quad W^i_{h(l)} = V_l W^i_h,$$

In view of (1.5) and (1.6). We observe that projective deviation tensor W^i_h and the Projective tensor W^i_{kh} are recurrent.

The projective curvature tensor satisfied the identity by Kumar, Shukla and Tripathi[7]:

$$(1.7)\quad W^i_{hjk(l)} + W^i_{hkl(j)} + W^i_{hlj(k)} = 0.$$

Sinha and Singh [5] have also defined that an F_n is called projective recurrent of second order, If the Weyl's projective curvature tensor satisfies

$$(1.8)\quad W^i_{jkh(l)(m)} = U_{lm} W^i_{jkh},$$

whereU_{lm} is a recurrence tensor. Transvecting (1.8) successively by $\dot{x}^j$and $\dot{x}^k$, we get

$$(1.9)\quad W^i_{kh(l)(m)} = U_{lm} W^i_{kh},$$

and

$$(1.10)\quad W^i_{h(l)(m)} = U_{lm} W^i_h.$$

Accordingly, we can state that Projective deviation tensor and projective tensor satisfies the second order recurrent condition, if so is Weyl's curvature tensor.

The recurrent curvature tensor H^i_{jkh}satisfies the relation Sinha and Singh[6]:

(1.11) $H^i_{jkh(l)} = V_l H^i_{jkh}$

Where V_l is recurrence vector. Transvecting (1.12) successively by $\dot{x}^j$ and $\dot{x}^k$ and, we get

(1.12) $H^i_{kh(l)} = V_l H^i_{kh}$
and

(1.13) $H^i_{h(l)} = V_l H^i_h,$

The curvature tensor field of second order satisfies the relations Sinha and Singh[6]:

(1.14) $H^i_{jkh(l)(m)} = K_{lm} H^i_{jkh},$

where $K_{lm} = V_m + V_{l(m)}$ is the recurrence tensor. Transvecting successively by $\dot{x}^j$and $\dot{x}^k$, we have

(1.15) $H^i_{kh(l)(m)} = K_{lm} H^i_{kh},$

and

(1.16) $H^i_{h(l)(m)} = K_{lm} H^i_h,$

2. DECOMPOSITION OF PROJECTIVE CURVATURE TENSOR IN RECURRENT FINSLER SPACE ($WR - F_n$)

Let us consider the projective curvature tensor W^i_{jkh} in the form

(2.1) $W^i_{jkh} = Y^i_j B_{kh},$

whereY^i_j is non zero tensor and B_{kh} is skew symmetric decomposition tensor.

The space equipped with such decomposition of projective curvature tensor with recurrent Finsler space is called decomposition of Projective curvature tensor in recurrent Finsler space and we denote it by $WR - F_n$.

Differentiating (2.1) covariantly with respect to x^l in the sense of Berwald's, we get

(2.2) $W^i_{jkh(l)} = Y^i_{J(l)} B_{kh} + B_{kh(l)} Y^i_{J.}$

Using the equation (1.4), (2.1) in (2.2), we get

(2.3) $W^i_{jkh(l)} = \beta_l Y^i_J B_{kh} + B_{kh(l)} Y^i_{J.}$

Where

(2.4) $Y^i_{J(l)} = \beta_l Y^i_{J.}$

From equation (2.1) and equation (2.3), we get

(2.5) $B_{kh(l)} = (V_l - \beta_l) B_{kh}.$

Let us assume that $(V_l \neq \beta_l)$ then the equation (2.5) may be written as

(2.6) $B_{kh(l)} = \gamma_l B_{kh},$

where $\gamma_l = (V_l - \beta_l)$.

Conversely,

If the above equation (2.6) is true then (2.3) yield

(2.7) $V_l B_{kh} = (\beta_l + \gamma_l) B_{kh}$

Accordingly, we have the

Theorem 2.1: In $WR - F_n$, the necessary and sufficient condition for the decomposition tensor B_{kh} to be recurrent is that the recurrent vector V_l is not equal to recurrent vector β_l.

Let us assume that the vector V_l is equal to recurrent vector β_l such that

(2.8) $V_l = \beta_l$

In view of equation (2.8), equation (2.6) immediately reduces to

(2.9) $B_{kh(l)} = 0.$

Using equation (2.9) in (2.3), we have

(2.10) $W^i_{jkh(l)} = Y^i_{J(l)} B_{kh}$

or $W^i_{jkh(l)} = \beta_l Y^i_J B_{kh}$

Adding the expressions obtained by cyclic change of (2.10) with respect to the indices $k,\ h$and l, we have

(2.11) $W^i_{jkh(l)} + W^i_{jhl(k)} + W^i_{jlk(h)}$

$= Y^i_j(\beta_l B_{kh} + \beta_k B_{hl} + \beta_h B_{lk}).$

In view of (1.4) equation (2.11) reduces to

(2.13) $Y^i_j(\beta_l B_{kh} + \beta_k B_{hl} + \beta_h B_{lk}) = 0.$

Since Y^i_j is non zero tensor, it implies

(2.13) $\beta_l B_{kh} + \beta_k B_{hl} + \beta_h B_{lk} = 0$

or $V_l B_{kh} + V_k B_{hl} + V_h B_{lk} = 0$

Accordingly state:

Theorem 2.2: In $WR - F_n$, under the decomposition (2.1), if the vector V_l is equal to β_l, the decomposition tensor satisfies the following identity (2.13).

Differentiating (2.10) covariantly with respect to x^m in the sense of Berwald's and using (2.9), we get

(2.14) $W^i_{jkh(l)(m)} = \beta_{l(m)} Y^i_j B_{kh} + \beta_l Y^i_{J(m)} B_{kh}$

In view of (2.4) the above equation may be written as

(2.15) $W^i_{jkh(l)(m)} = (\beta_{l(m)} + \beta_l \beta_m) Y^i_j B_{kh}$,

Using equation (1.10) and (2.1), we get

(2.16) $U_{lm} Y^i_j B_{kh} = (\beta_{l(m)} + \beta_l \beta_m) Y^i_j B_{kh}$,

From (2.15) and (2.16), we have

(2.17) $U_{lm} = (\beta_{l(m)} + \beta_l \beta_m)$

Thus we conclude that

Theorem 2.3 In $WR - F_n$, under the decomposition (2.1), if the vector V_l is equal to β_l for which recurrence vector field β_l satisfies the condition $\beta_{l(m)} + \beta_l \beta_m \neq 0$.

Interchanging the indices l and m in (2.15) and subtracting the equation thus obtained to (2.15), we have

(2.18) $W^i_{jkh(l)(m)} - W^i_{jkh(m)(l)}$

$$= (U_{lm} - U_{m\,l}) Y^i_j B_{kh} B_{kh}.$$

or

$$W^i_{jkh(l)(m)} - W^i_{jkh(m)(l)} = (\beta_{l(m)} - \beta_{m(l)}) Y^i_j B_{kh}.$$

Accordingly we state:

Corollary 2.1: In $WR - F_n$, Under the decomposition (2.1) if the vector V_l is equal to β_l, the projective curvature tensor satisfies the following identity (2.18).

Differentiating (2.6) covariantly with respect to x^m in the sense of Berwald's, we get

(2.19) $B_{kh(l)(m)} = \gamma_{l(m)} B_{kh} + \gamma_l B_{kh(m)}$

$$= (V_{l(m)} - \beta_{l(m)}) B_{kh} + (V_l - \beta_l) B_{kh(m)}$$

In view of (2.6), the equation (2.19) may be written as

(2.20) $B_{kh(l)(m)} = (V_{l(m)} - \beta_{l(m)}) B_{kh} + \gamma_l (V_l - \beta_l) B_{kh}$

or

(2.21) $B_{kh(l)(m)} = (V_{l(m)} - \beta_{l(m)}) B_{kh} +$

$$(V_m - \beta_m)(V_l - \beta_l) B_{kh}$$

(2.22) $B_{kh(l)(m)} = (V_{l(m)} - \beta_{l(m)} + V_m V_l$

$$- V_m \beta_l - V_l \beta_m + \beta_l \beta_m) B_{kh}$$

Theorem 2.4: In $WR - F_n$, Under the decomposition (2.1), the second order covariant derivative of decomposition tensor B_{kh} satisfies the relation (2.22).

In view of equation (2.8), equation (2.22) immediately reduces to

(2.23) $B_{kh(l)(m)} = 0$

Corollary 2.2: $WR - F_n$, the second order covariant derivative of decomposition tensor B_{kh} vanish, If the vector V_l is equal to β_l.

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