A Method of Curve Fitting by Recurrent Fractal Interpolation

Bhagwati Prasad
Department of Mathematics
Jaypee Institute of Information Technology
A-10, Sector-62, Noida
201307, India

Bani Singh
Department of Mathematics
Jaypee Institute of Information Technology
A-10, Sector-62, Noida
201307, India

Kuldip Katiyar
Department of Mathematics
Jaypee Institute of Information Technology
A-10, Sector-62, Noida
201307, India

ABSTRACT
The real world objects are too irregular to be modeled with the help of traditional interpolation methods. M. F. Barnsley in 1986 proposed the concept of fractal interpolation function (FIF) using iterated function systems (IFS) to describe such real world data. In many cases these data sets represent a curve rather than a function i.e. the data points are not linearly ordered with their abscissa and self affinity is not satisfied in the whole range. The recurrent fractal interpolation function (RFIF) has a role to play in such cases. The purpose of this paper is to apply recurrent fractal interpolation function to fit the piecewise self affine data.

Keywords
Fractal interpolation, Recurrent fractal interpolation function, Piecewise self affine function, curve fitting.

1. INTRODUCTION
Most of the natural objects such as coastline, fracture surface etc are generally rough, unsmoothed in nature and have some degree of self-similarity. In 1975, Mandelbrot [10] introduced fractals to study such objects. Thereafter, fractals are widely studied by a number of authors. Barnsley [1-4], Dalla and Drakopoulos [6], Hutchinson [9], Mandelbrot [10], Manousopoulos et al. [14], Navascues et al [15], Mazel et al. [11-13], Prasad and Katiyar [16], Singh et al. [18] and many others have enriched the theory of fractals by identifying the diverse domain of applications of fractals. Barnsely [1] proposed the method of fractal interpolation in the year 1986 and studied the continuity and the fractal dimension of fractal interpolation function. Fractal interpolation method, different from the traditional interpolation method, is a numerical method in the fractal geometry which uses the concepts of iterated function system (IFS). The FIFs are very efficient to approximate unsmoothed curves. Such curves can be described by means of a fractal interpolation functions due to the unique characteristic applied in unsmoothed curve fitting. The graphs of fractal interpolating functions can also be used to approximate image components of many natural objects such as the profiles of mountain ranges, the tops of clouds, stalactite-hung roofs of caves and horizons over forests. This technique is widely used in simulation, modelling and computer graphics. The fractal interpolation functions (FIF) have opened a new research field in the approximation theory of functions. In the recent years, it had gained an appreciable attention of researchers working in these areas. There are many papers in the literature about the applications and properties of such functions, see for instance [1]-[5], [8], [17], [19] and references thereof.

Barnsley et al. [2] generalized FIF in the form of recurrent fractal interpolation function (RFIF) to fit data set which is piecewise self-affine. In practice, we find there are many cases where the data set defines a curve rather than a function. Therefore one can not apply fractal interpolation directly, for example, in modeling coastlines or plants. A number of methods are used in the literature for fractal fitting of self affine data sets. The methods based on generalizations to higher dimensions are introduced in [5], [11] and [13]. Uemura et al [19] proposed a method based on index coordinates. Various combinations of IFS models and free form curves are proposed by Guerin et al. [7-8]. Manpusopoulos et al. [14] introduced a method in which FIF is obtained from the transformed data set. In many practical situations, the interpolating curve may not be self affine in the entire range. Piecewise self affinity can be easily explored in such cases. This motivates us to introduce a new algorithm for modeling curves which are piecewise self-affine.

2. PRELIMINARIES
In this section we present the basic definitions and concepts required for our study.

Definition 2.1 [3]. Let (X, d) be a metric space. A transformation $w: X \to X$ is said to be Lipschitz with Lipschitz constant $s \in R$ iff $d(w(x), w(y)) \leq s\, d(x, y)$ for all $x, y \in X$. A transformation $f: X \to X$ is called contractive iff it is Lipschitz with Lipschitz constant $s \in [0, 1)$. A Lipschitz constant $s \in [0, 1)$ is also called a contraction factor.

Definition 2.2 [3]. A hyperbolic iterated function system (IFS) consists of a complete metric space (X, d) together with a finite set of contraction mappings $w_n: X \to X$, with respective contractivity factors s_n for $n = 1, 2, ..., N$. This IFS is represented by $\{X; w_n: n = 1, 2, ..., N\}$ with contractivity factor $s = \max\{s_n: n = 1, 2, ..., N\}$.

Definition 2.3 [3]. A recurrent iterated function system (RIFS) consists of an IFS $\{X; w_n: n = 1, 2, ..., N\}$ together with a matrix $\{p_{mn} \in [0, 1]: m, n = 1, 2, ..., N\}$, such that

(*i*) $p_{m\,1} + p_{m\,2} + p_{m\,3} + \cdots + p_{m\,N} = 1$ for $m = 1, 2, ..., N$;

(ii) for any m and n, there exists a finite sequence of integers $k, l, ..., q \in \{1, 2, ..., N\}$ so that

$$p_{m\,k}\ p_{k\,l}\ \ldots\ p_{q\,n} > 0.$$

The RIFS is represented by $\{X, w_n, p_{m\,n}, m, n = 1, 2, ..., N\}$.

Definition 2.4 [3]. Let (X, d) be a metric space and $H(X)$ denote the nonempty compact subsets of X. Then the Hausdorff metric h in $H(X)$ is defined as

$h(A, B) = \max \{d(A, B), d(B, A)\}$ for all $A, B \in H(X)$,

where $d(A, B) = \max(\min(d(a, b): b \in B): a \in A)$.

We now state a lemma of Barsnley [3] which guarantees a contraction map in $\{H(X), h\}$ out of a contraction map on (X, d).

Lemma 2.5 [3]. Let $w: X \to X$ be a contraction on a metric space (X, d) with contractivity factor s. Then $w: H(X) \to H(X)$ defined by

$$w(B) = \{w(x) : x \in B\} \quad \forall B \in H(X)$$

is a contraction on $\{H(X), h\}$ with contractivity factor s.

The following theorem ensuring the existence of a unique fixed point (also called an attractor) of the IFS, is fundamental to our results.

Theorem 2.6 [2]. Let $\{X; w_n, n = 1, 2, ..., N\}$ be a hyperbolic iterated function system with contractivity factor s. Then the transformation W: $H(X) \to H(X)$ defined by $W(B) = \bigcup_{n=1}^{N} w_n(B)$ for all $B \in H(X)$ is a contraction mapping on the complete metric space $(H(X), h)$ with contractivity factor s. That is, $h(W(B), W(C)) \le s\, h(B, C)$ for all $B, C \in H(X)$. Its unique fixed point, $A \in H(X)$ obeys $A = W(A) = \cup_{n=1}^{N} w_n(A)$ and is given by $A = \lim_{n\to\infty} W^n(B)$ for any $B \in H(X)$.

2.1 Recurrent Fractal Interpolation Function

Let $\Delta_1 = \{(x_n, y_n) \in I\times R: n = 0, 1, \ldots, N\}$ be the data set, where $I = [x_0, x_N] \subset R$ and $x_0 < x_1 < \ldots < x_N$. The interpolation points divide I into N intervals $I_n = [x_{n-1}, x_n]$, $n = 1, 2, \ldots, N$. Let $\Delta_2 = \{(\tilde{x}_m, \tilde{y}_m) : m = 0, 1, \ldots, M\} \subset \Delta_1$, divides I into M sections $S_m = [\tilde{x}_{m-1}, \tilde{x}_m]$, $m = 1, 2, \ldots, M < N$ such that $x_0 = \tilde{x}_0 < \tilde{x}_1 < \tilde{x}_2 < \ldots < \tilde{x}_M = x_N$ and there is at least one n such that $\tilde{x}_m < x_n < \tilde{x}_{m+1}$ for every m. We associate each interpolation interval to a pair of data points called address points. Let the nth interval I_n is associated with the data points $(x'_{n,1}, y'_{n,1})$ and $(x'_{n,2}, y'_{n,2})$ for $n = 1, 2, ..., N$, where $(x'_{n,k}, y'_{n,k}) = (\tilde{x}_m, \tilde{y}_m)$ for $m = 1, 2, \ldots M$ and $k = 1, 2$. Each pair of address points defines a section $[x'_{n,1}, x'_{n,2}]$ with $x'_{n,1} < x'_{n,2}$ for $n = 1, 2, \ldots N$. Let $x_n - x_{n-1} = \delta_n, n = 1, 2, \ldots, N$ and $\tilde{x}_m - \tilde{x}_{m-1} = \psi_m$, $m = 1, 2, \ldots, M$ such that $\psi_m < \delta_n$ for $n = 1, 2, \ldots N$, $m = 1, 2, \ldots M$.

Now we define $w_n: I\times R \to R^2$, $n = 1, 2, ..., N$ in the following manner:

$$w_n\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_n & 0 \\ c_n & d_n \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_n \\ f_n \end{pmatrix} \tag{2.1}$$

The constants are chosen such that each map w_n is constrained to map the endpoints of the section S_m to the endpoints of the interval I_n. That is,

$$w_n\begin{pmatrix} x'_{n,1} \\ y'_{n,1} \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix},$$

$$w_n\begin{pmatrix} x'_{n,2} \\ y'_{n,2} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} \tag{2.2}$$

From which we obtain

$$a_n = \frac{x_n - x_{n-1}}{x'_{n,2} - x'_{n,1}} \qquad e_n = \frac{x'_{n,2}x_{n-1} - x'_{n,1}x_n}{x'_{n,2} - x'_{n,1}}$$

$$f_n = \frac{x'_{n,2}y_{n-1} - x'_{n,1}y_n}{x'_{n,2} - x'_{n,1}} - d_n\,\frac{x'_{n,2}y'_{n,1} - x'_{n,1}y'_{n,2}}{x'_{n,2} - x'_{n,1}} \tag{2.3}$$

for every $n = 1, 2, 3, \ldots, N$. The real numbers a_n, b_n, c_n, e_n are completely determined by the interpolation and address points, while the d_n (called vertical scaling or contractivity factors) are free parameters of the transformations w_n satisfying $|d_n| < 1$, so that the transformations w_n are contractive with respect to a metric equivalent to Euclidean metric.

Moreover, let $W(B) = \cup_{n=1}^{N} w_n(B^{[n]})$, where $B \in H(R^2)$ and $B^{[n]} = \{(x, y) \in B : x'_{n,1} \le x \le x'_{n,2}\}$, for $n = 1, 2, ..., N$. The unique set $G \equiv A = \lim_{k\to\infty} W^k(B)$ for every starting set $B \in H(R^2)$, is the graph of a continuous function $f: I \to R$ that passes through the interpolation points (x_n, y_n), for all $n = 0, 1, ..., N$ (see[2, 3, 12]). We call such a function a recurrent fractal interpolation function (RFIF) or piecewise self affine fractal interpolation function.

3. PIECEWISE SELF AFFINE FRACTAL INTERPOLATING CURVE

We extend the idea of fractal interpolating curve given by Manousopoulos et al. [14] for piecewise self affine maps. We apply Mazel and Hayes [12] algorithm to calculate the vertical scaling factors and identify the best matching section for a given interpolating points. Let $\Delta = \{(u_n, v_n) \in I\times R: n = 0, 1, \ldots, N\}$ be the given set of data point.

Transform the given set of data Δ to (x_n, y_n), $n = 1, 2, 3, \ldots, N$ using method given by Manousopoulos et.al. [14].

First label it as $\{(u_{J(i)}, v_{J(i)}) \in R^2 : i = 0, 1, 2, \ldots, L\}$, where the labeling function $J:\{0, 1, ..., L\} \to \{0, 1, ..., N\}$ defines the indices of the interpolation points. After this apply the transform $T_1(u_n, v_n) = (x_n, y_n), n = 0, 1, 2, \ldots, N$, where

$$x_n = u_0 + \sum_{j=1}^{n}(|u_j - u_{j-1}| + \varepsilon) = u_{n-1} + (|u_j - u_{j-1}| + \varepsilon)$$
$$y_n = v_n,$$

and $\varepsilon > 0$ is an arbitrary constant necessary when all points in an interpolation interval have equal x-coordinates, i.e. $u_n = u_{n-1}$ for every $n = J(i)+1, ..., J(i+1)$ and some $i \in \{0, 1, ..., L\}$. Otherwise, we set $\varepsilon = 0$.

Now apply the following algorithm to select the best matching section for every interpolation interval I_n and corresponding contractivity factors d_n.

1. Choose δ_n, ψ_m with $\psi_m < \delta_n$.
2. For each interpolating interval n do:
   a) for each section m do:
      (i) Use least square method to compute the contraction factor for the map associated with the nth interval and mth section (see Mazel and Hzyes [12]).
      (ii) If $|d| \geq 1$, go to (vi).
      (iii) Form the map $w\begin{pmatrix} x \\ y \end{pmatrix}$ associated with the nth interval and mth section.
      (iv) Apply $w\begin{pmatrix} x \\ y \end{pmatrix}$ to function values between mth section and call this $\hat{H}$.
      (v) Compute the distance h_m between $\hat{H}$ and the function values over the support $[x_{n-1}, x_n]$ i.e. $h_m = h\left(\hat{H}, H|_{x\in[\tilde{x}_{m-1},\tilde{x}_m]}\right)$.
      (vi) Next section.
   b) Find the m for which h, is a minimum.
   c) Store the map parameters for the nth interval associated mth section as the map parameters for the nth interval.
   d) Next interval.

Now, apply an affine RFIF to create an RIFS whose attractor is the graph G' of a function that interpolates the points $(u_{J(n)}, v_{J(n)})$ $n = 0, 1,..., N$.

The final step is to apply a transformation to G' in order to obtain the graph G of a curve that interpolates the initial points $\{(u_n, v_n): n = 0, 1, ..., N\}$. Let $(x, y) \in G'$ be a point of the attractor. We apply the transformation $T_2: G' \to G$ with $(x, y) \to (u, v)$, where

$$u = u_{n-1} + (u_n - u_{n-1})\left(\frac{x - x_{n-1}}{x_m - x_{n-1}}\right), \quad x \in [x_{n-1} - x_n]$$

$$v = y.$$

Fractal interpolated curves (FICs) of this kind are depicted in Fig. 1, which are constructed on a simple, manually selected set of 10 interpolation points. Specifically, we select the following data points

{(3, 1), (2, 2), (1, 4), (0, 3), (-1, 3), (-2, 1), (-1, -1), (0, -2), (2, -1), (3.5, -0.5)}

The contractivity factors d_n have been calculated using least square method.

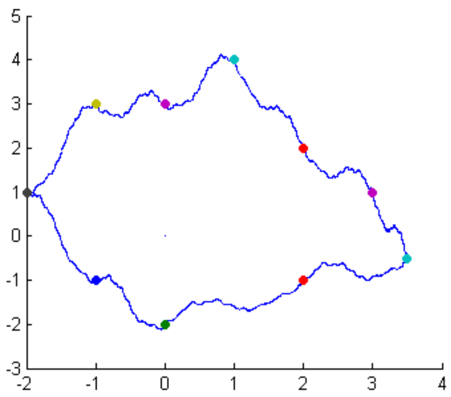
Fig.1 shows the graphs of piecewise self affine curves as an attractor of RIFS. In figure 1(a) we use fractal interpolation function with contractivity factors $d_n = 0.1$ for all n, to fit the graph as used by Manousopoulos et al. [14].

In Fig. 1(b) – (d), we consider the partition Δ_2 as follows:

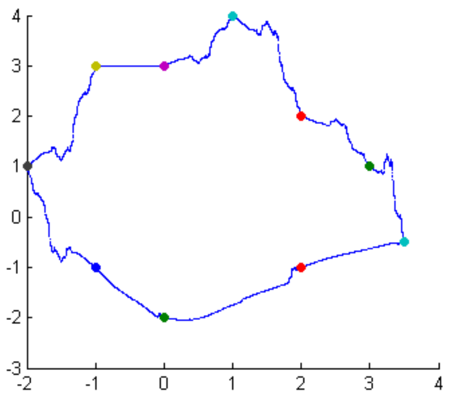
Fig. 1(b) - $\{[x_1, x_5], [x_5, x_7], [x_7, x_{11}]\}$,

Fig. 1(c) - $\{[x_1, x_4], [x_4, x_9], [x_9, x_{11}]\}$,

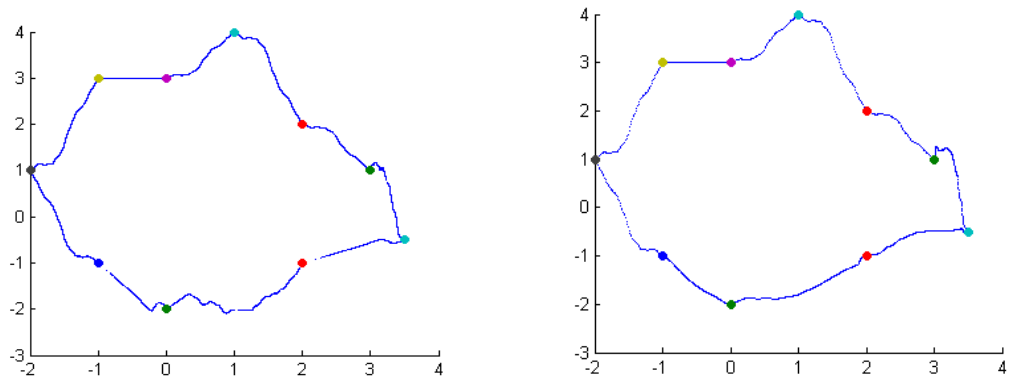
Fig. 1(d) - $\{[x_1, x_4], [x_4, x_{11}]\}$,



(*a*)



(*b*)



(*c*) (*d*)

Fig 1: Fractal interpolated curves

4. CONCLUSIONS

While modeling various problems natural or otherwise, it is not always possible to have the self affinity condition fully satisfied in the entire range of the data set. In such cases our approach of recurrent fractal interpolating function (RFIF) is better than the usual fractal interpolating function (FIF). We explore the sub intervals of the data sets showing self affinity to use the proposed algorithm to fit the data set.

5. REFERENCES

[1] Barnsley, M. F. 1986. Fractal functions and interpolation, Constr. Approx. 2, 303-329.

[2] Barnsley, M. F., John H. Elton, D. and Hardin, P. 1989. Recurrent iterated function systems in Fractal Approximation, Constr. Approx. 5, 3–31.

[3] Barnsley, M. F. 1993. Fractals Everywhere, 2nd edition, Revised with the assistance of and a foreword by Hawley Rising, III. Boston MA, Academic Press Professional.

[4] Barnsley, M. F., Hutchinson, J. E. and Stenflo, O. 2008. Fractals with partial self similarity, Adv. Math 218(6), 2051-88.

[5] Cochran, W. O., Hart, J. C. and Flynn, P. J. 1998. On approximating rough curves with fractal functions. Proc. Graphics Interface. 1, 65–72.

[6] Dalla, L. and Drakopoulos, V. 1999. On the parameter identification problem in the plane and the polar fractal interpolation functions. J. Approx. Theory 101, 290–303.

[7] Guerin, E., Tosan, E. and Baskurt, A. 2000. Fractal coding of shapes based on a projected IFS model. In: ICIP, 2, IEEE Computer Society Press, Los Alamitos, 203–206.

[8] Guerin, E., Tosan, E. and Baskurt, A. 2001. A fractal approximation of curves. Fractals 9, 95–103.

[9] Hutchinson, J. E. 1981. Fractals and self similarity, Indiana Univ. Math. J. 30, 713-747.

[10] Mandelbrot, B. B. 1977. The Fractal Geometry of Nature, Updated and Augmented, International Business Machines, Thomas J. Watson Research Center, H. Freeman And Company, New York.

[11] Mazel, D. S. and Hayes, M. H. 1991. Hidden-variable fractal interpolation of discrete sequences. Proc. Int. Conf. ASSP.1, 3393–3396

[12] Mazel, D. S. and Hayes, M. H. 1992. Using iterated function systems to model discrete sequences. IEEE Trans. Signal Process. 40, 1724–1734

[13] Mazel, D. S. 1994. Representation of discrete sequences with three-dimensional iterated function systems. IEEE Trans. Signal Processing 42, 3269–3271.

[14] Manousopoulos, P., Drakopoulos, V. and Theoharis, T. 2008. Curve fitting by fractal interpolation. Trans. Comput. Sci. 1, 85–103.

[15] Navascues, M. A. and Sebastian, M. V. 2004. Fitting curves by fractal interpolation: An application to the quantification of cognitive brain processes. In: Novak, M.M. (ed.) Thinking in patterns: Fractals and related phenomena in nature, World Scientific, Singapore, 143–154.

[16] Prasad, B. and Katiyar, K. 2011. Fractals via ishikawa iteration, CCIS, Springer Berlin Heidelberg, 140(2), 197-203.

[17] Priceand, J. R. and Hayes, M. H., 1998. Resampling and reconstructing with fractal interpolation functions, IEEE Signal Process. Letters, 5, 228-230.

[18] Singh, S. L., Prasad, B. and Kumar, A. 2009. Fractals via iterated functions and multifunctions. Chaos, Solitons and Fractals 39, 1224-1231.

[19] Uemura, S., Haseyama, M., and Kitajima, H. 2002. Efficient contour shape description by using fractal interpolation functions. IEEE Proc. ICIP. 1, 485–488.