Inversion of Fractional Hankel Transform in the Zemanian Space

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ABSTRACT

The fractional Hankel transform which is a generalization of the Hankel transform has many applications. In this paper we have derived inversion theorem for the generalized Fractional Hankel transform so that the transform can be used in solving partial differential equations or boundary value problems.

Keywords

Fourier transform, Hankel transform, fractional Hankel transform.

1. INTRODUCTION

Namias [4] introduced the concept of Fourier transform of fractional order by the method of eigen values and opened the path of defining number of fractional integral transforms. Observing the applications of fractional integral transform in quantum mechanics [4], optics [2], signal processing [3], Hankel transform is also generalized to its fractional version. Namias [5] himself had presented the formula for fractional Hankel transform as

$$H_\alpha f(x) = \int_0^\infty K_\alpha(x,y) f(x)dx,$$

where

$$K_\alpha(x,y) = \frac{e^{i(v+1)\left[\frac{\pi}{2}-\frac{\alpha}{2}\right]}}{\sin\frac{\alpha}{2}} \exp\left(\frac{ix^2}{2}\cot\frac{\alpha}{2}\right) y \exp\left(\frac{iy^2}{2}\cot\frac{\alpha}{2}\right) J_v\left(\frac{xy}{\sin\frac{\alpha}{2}}\right)$$

but it is the generalization of the conventional Hankel transform,

$$F(y) = \int_0^\infty f(x) J_v(xy) x dx .$$

For suitable function f, Zemanian [1] defined the Hankel transform of f of Bessel order μ by

$$(hf)(y) = \int_0^\infty f(x)\sqrt{xy}\, J_\mu(xy)dx$$

Motivated by above, in this paper we have extended fractional Hankel transform which is the generalization of the Hankel transform $\int_0^\infty f(x)\sqrt{xy}\, J_\mu(xy)dx$,

for the parameter α in generalized sense.

For $\alpha = \pi$ the fractional Hankel transform reduces to the above Hankel transform in [1].

The paper is organized as follows. In section 2 we have defined a testing function, section 3 gives the inversion theorem with two lemmas, section 4 gives uniqueness theorem, section 5 concludes the paper.

2. THE TESTING FUNCTION SPACE E

An infinitely differentiable complex valued function ϕ on R^n belongs to E (R^n) or E if for each compact set K$\subset S_a$,

Where $S_a = \{x \in R^n, |x| \le a, a > 0\}$, k$\in N^n$,

$$\gamma_{K,k}(\phi) = \sup_{x\in K} \left|D^k \phi(x)\right| < \infty .$$

Clearly E is complete and so a Frechet space.

2.1 The fractional Hankel transform on E' ;

It is easily seen that for each x$\in R^n$ and 0<$\alpha < 2\pi$ the function $K_\alpha(x,y)$ belongs to E as a function of x.

Hence the fractional Hankel transform of f$\in E'$ can be defined by ,

$$[h_\alpha f(x)](y) = h_\alpha(y) = \langle f(x), K_\alpha(x,y)\rangle \qquad (2.1.1)$$

Where $K_\alpha(x,y) =$

$$A_{v,\alpha} \exp\left(-\frac{i}{2}\left((x^2+y^2)\cot\frac{\alpha}{2}\right)\right)\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}J_v\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)\right)$$

$A_{v,\alpha} =$

$$\left|\sin\frac{\alpha}{2}\right|^{-1/2} \exp\left(i\left(\frac{\pi}{2}\alpha-\frac{\hat{\alpha}}{2}\right)(v+1)\right), \hat{\alpha} = \operatorname{sgn}\alpha,$$

then the right hand side of (2.1) has a meaning as the application of $f \in E'$ to $K_\alpha(x,y) \in E.$

3 INVERSION THEOREM

Let $f \in E'(R)$, $0 \le \alpha \le \pi$ and $\operatorname{supp} f \subset S_a$ where $S_a = \{x : x \in R, |x| < a, a > 0\}$ and let $h_\alpha(y)$ be the generalized fractional Hankel transformation of f defined by

$$[h_\alpha f(x)](y) = h_\alpha(y) = \langle f(x), K_\alpha(x,y)\rangle.$$

Then for each $\phi \in E$, we have

$$\langle f(x), \phi(x)\rangle = \left\langle \int_0^\infty \overline{K_\alpha(x,y)} h_\alpha(y)dy, \phi(x)\right\rangle,$$

where

$$\overline{K_\alpha(x,y)} = \left|\sin\frac{\alpha}{2}\right|^{-1} \exp\left(i\left(\frac{\alpha}{2}-\frac{\pi}{2}\hat{\alpha}\right)(v+1)\right)$$

$$\exp\left(\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}\right) J_v\left(xy \;\; 1\Big/\left|\sin\alpha/2\right|\right)(xy)^{\frac{1}{2}}$$

Proof:--

To prove the inversion theorem, we state the following lemmas to be used in the sequel.

Lemma 1:--

Let $[h_\alpha f(x)](y) = h_\alpha(y)$ for $0 < \alpha < \pi$ and supp S_a for $\phi(x) \in E$,

$$\psi(y) = \int_0^\infty \overline{K_\alpha(x,y)}\phi(x)dx.$$

Then for any fixed number r, $0 < r < \infty$

$$\int_0^r \psi(y)\langle f(\varsigma), K_\alpha(\varsigma,y)\rangle d\tau = \left\langle f(\varsigma), \int_0^r \psi(y)K_\alpha(\varsigma,y)d\tau\right\rangle \qquad (3.1)$$

where y= $\sigma + i\tau \in C^n$ and ς is restricted to a compact subset of R.

Proof:

The case $\phi(x) = 0$ is trivial, for $\phi(x) \neq 0$ it can be easily seen that $\int_0^r \psi(y)K_\alpha(\varsigma,y)d\tau,$ where y= $\sigma + i\tau$ *is* C^∞ - function of ς and it belongs to E. Hence the right hand side of (3.1) is meaningful.

To prove the equality, we construct the Riemann-sum for this integral and write,

$$\int_0^r \psi(y)\langle f(\varsigma), K_\alpha(\varsigma,y)\rangle d\tau =$$

$$\lim_{m\to\infty}\left\langle f(\varsigma), \sum_{n=0}^{m-1} K_\alpha(\varsigma, \sigma + i\tau_{n,m}\psi(\sigma + i\tau_{n,m})\Delta\tau_{n,m}\right\rangle$$

We show that the last summation converges in E to the integral on the right hand side of (3.1). Consider

$$\gamma_{k,K}\left\{\begin{array}{l}\sum_{n=0}^{m-1} K_\alpha(\varsigma, \sigma + i\tau_{n,m})\psi(\sigma + i\tau_{n,m})\Delta\tau_{n,m} \\ -\int_0^r \psi(y)K_\alpha(\varsigma,y)d\tau\end{array}\right\}$$

$$= \sup_{\varsigma\in K}\left\{\begin{array}{l}\sum_{n=0}^{m-1} D_\varsigma^k K_\alpha(\varsigma, \sigma + i\tau_{n,m})\Delta\tau_{n,m} \\ -\int_0^r \psi(y)K_\alpha(\varsigma,y)d\tau\end{array}\right\}$$

Carrying the operator D_ς^k within the integral and summation signs, this is easily justified. We get

$$\lim_{m\to\infty}\sum_{n=0}^{m-1} D_\varsigma^k K_\alpha(\varsigma, \sigma + i\tau_{n,m})\psi(\sigma + i\tau_{n,m})\Delta\tau_{n,m}$$

$$= \int_0^r \psi(y)K_\alpha(\varsigma,y)d\tau, \quad \forall \zeta \in K.$$

It thus follows that for every m, the summation is a member of E and it converges in E to the integral on the right hand side of (3.1).

Hence the proof.

Lemma 2:-

For $\phi(x) \in E$, set $\psi(y)$ as in Lemma 1 above for $y \in C$, ς is restricted to a compact subset of R then,

$$M_r(\varsigma) = \int_0^r K_\alpha(\varsigma, y) \int_0^\infty \phi(x) \overline{K_\alpha(x, y)} dX d\tau \qquad (3.2)$$

Converges in E to $\phi(\varsigma)$ as $r \to \infty$.

Proof:-

We shall show that $M_r(\varsigma) \to \phi(\varsigma)$ in E as $r \to \infty$. That is to show,

$$\gamma_{K,k}\left[M_r(\varsigma) - \phi(\varsigma)\right] = \sup_{\varsigma \in K}\left\{D_\varsigma^k\left[M_r(\varsigma) - \phi(\varsigma)\right]\right\} \to 0 \quad \text{as } r \to \infty$$

We note that for k=0,

$$\int_0^r K_\alpha(\varsigma, y)\left(\int_0^\infty \phi(x)\overline{K_\alpha(x, y)} dX\right) d\tau = \phi(\varsigma)$$

That is to say that $\lim_{r \to \infty} M_r(\varsigma) = \phi(\varsigma)$

Since the integrand is a C^∞- function of ς and $\phi \in E$, we can repeatedly differentiate under the integral sign in (3.2) and the integrals are uniformly convergent. We have

$$\int_0^r D_\varsigma^k K_\alpha(\varsigma, y)\left(\int_0^\infty \phi(x)\overline{K_\alpha(x, y)} dx\right) d\tau = \phi(\varsigma)$$

for all $\varsigma \in K$,

Hence the claim

.

Proof of inversion Theorem:

Let $\phi(x) \in E$. We shall show that

$$\left\langle \int_0^\infty \overline{K_\alpha(x, y)} H_\alpha(y) d\tau, \phi(x) \right\rangle$$

tends to $\langle f(x), \phi(x) \rangle$ as $r \to \infty$ (3.3)

From the analyticity of $H_\alpha(y)$ on C and the fact that $\phi(x)$ has a compact support in R, it follows that the left side expression in (3.3) is merely a repeated integral with respect to x and y and the integral in (3.3) is a continuous function of x as the closed bounded domain of the integration.

Therefore we write (3.3) as,

$$\int_0^\infty \phi(x) \int_0^r \overline{K_\alpha(x, y)} H_\alpha(y) d\tau dx = \int_0^r \langle f(\varsigma), K_\alpha(\varsigma, y) \rangle \psi(y) d\tau$$

Since $\phi(x)$ is of compact support, and the integrand is a continuous function of (x, y) the order of integration may be changed. The change in the order of integration is justified, where $\psi(y) = \int_0^\infty \phi(x)\overline{K_\alpha(x, y)} dx$,

this yields ,

$$\int_0^r \langle f(\varsigma), K_\alpha(\varsigma, y) \rangle \psi(y) d\tau = \left\langle f(\varsigma), \int_0^\infty K_\alpha(\varsigma, y) \psi(y) d\tau \right\rangle \qquad (3.4)$$

Again by lemma 2, R.H.S. of equation (3.4) converges to $\langle f(\varsigma), \phi(\varsigma) \rangle$ as $r \to \infty$.

This completes the proof of the theorem.

4. Uniqueness **theorem**

If $[H_\alpha f(x)](y) = H_\alpha(y)$ and

$[H_\alpha g(x)](y) = G_\alpha(y)$ for $0 < \alpha \le \frac{\pi}{2}$

supp$f \subset S_a = \{x : x \in R, |x| \le a, \ a > 0\}$ and

supp$g \subset S_a = \{x : x \in R, |x| \le a, \ a > 0\}$

if $H_\alpha(y) = G_\alpha(y)$,

then $f = g$ in the sense of equality in E'.

Proof:

By inversion theorem

$$f - g = \int_0^\infty \overline{K_\alpha(x, y)}\left[H_\alpha(y) - G_\alpha(y)\right] dy$$

= 0, as $H_\alpha(y) = G_\alpha(y)$, thus $f = g$ in E',

this proves uniqueness.

5. CONCLUSION

We have given the inversion theorem for the generalized fractional Hankel transform with two lemmas. Uniqueness theorem is also given. Using inversion theorem, we can derive operational calculus which will be applicable to solve partial differential equations with boundary value problem.

6. REFERENCES

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