Unique Common Fixed Point Theorem in G- Metric Space via Rational Type Contractive Condition

Manjusha P.Gandhi
Yeshwantrao Chavan College of Engineering, Hingna Road Wanadongri, Nagpur-441110, India

Kavita B. Bajpai
Karmavir Dadasaheb Kannamwar College of Engineering, Nandanvan Nagpur, India

ABSTRACT
We prove a unique common fixed point theorem for three mappings in a G-metric space satisfying rational type contractive condition. Our result is generalization of result of P. L. Sanodia et al [6].

Keywords
Fixed point, Complete G- metric space, G-Cauchy sequence, rational contraction mapping.

1. INTRODUCTION
It is well known that the contractive type conditions play an important role in the study of fixed point theory. The first intrusting result on fixed point for contractive type mappings was established by Banach Caccioppoli in 1922. The Banach contraction principle has been generalized by many mathematicians Kannan [5], Chatterjea [1], Sehgal [7].
Further Hardy and Rogers [4], Ciric [3], Singh [8] define some new contraction conditions in metric space.
Mustafa in collaboration with Sims [10] introduced a new notation of generalized metric space called G- metric space in 2006. He proved many fixed point results for a self mapping in G- metric space under certain conditions. Many researchers W. Shatanawi [9], Z. Mustafa [11] proved fixed point theorems satisfying certain contractive conditions in G-metric space. P.L. Sanodia [6] et al proved a fixed point theorem for single mapping in G-metric space.
In the present work we prove a unique common fixed point result for three mappings in a complete G- metric space X under rational type contractive condition.
Now, we give preliminaries and basic definitions which are used through-out the paper.

Definition 1.1: Let X be a non empty set, and let $G: X \times X \times X \to R^+$ be a function satisfying the following properties:

(G_1) $G(x, y, z) = 0$ if $x = y = z$

(G_2) $0 < G(x, x, y)$ for all $x, y \in X$,with $x \neq y$

(G_3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x)$ (Symmetry in all three variables)

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality)

Then the function G is called a generalized metric space, or more specially a G- metric on X, and the pair (X, G) is called a G−metric space.

Definition 1.2: Let (X, G) be a G - metric space and let $\{x_n\}$ be a sequence of points of X , a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$, if $\lim_{n,m \to +\infty} G(x, x_n, x_m) = 0$, and we say that the sequence $\{x_n\}$ is G - convergent to x or $\{x_n\}$ G -converges to x .

Thus, $x_n \to x$ in a G - metric space (X, G) if for any $\in > 0$ there exists $k \in N$ such that $G(x, x_n, x_m) < \in$, for all $m, n \geq k$

Proposition 1.3: Let (X, G) be a G - metric space. Then the following are equivalent:

i) $\{x_n\}$ is G - convergent to x

ii) $G(x_n, x_n, x) \to 0$ as $n \to +\infty$

iii) $G(x_n, x, x) \to 0$ as $n \to +\infty$

iv) $G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$

Definition 1.4: Let (X, G) be a G - metric space. A sequence $\{x_n\}$ is called a G - Cauchy sequence if for any $\in > 0$ there exists $k \in N$ such that $G(x_n, x_m, x_l) < \in$ for all $m, n, l \geq k$, that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty.$

Proposition 1.5: Let (X, G) be a G - metric space .Then the following are equivalent:

i) The sequence $\{x_n\}$ is G - Cauchy;

ii) For any $\in > 0$ there exists $k \in N$ such that $G(x_n, x_m, x_m) < \in$ for all $m, n \geq k$

Proposition 1.6: A G - metric space (X, G) is called G -complete if every G -Cauchy sequence is G -convergent in (X, G) .

Proposition 1.7: Let (X,G) be a G-metric space. Then $f: X \to X$ is G-continuous at $x \in X$, if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.

The result is the generalization of the following result:

Theorem 1.8: Let X be a complete G-metric space. Suppose the map $T: X \to X$ satisfies

$$G(Tx,Ty,Tz) \le \alpha \frac{\max.\left\{\begin{matrix} G^2(x,Tx,Ty), G^2(y,Ty,Tz), \\ G^2(z,Tz,Tx) \end{matrix}\right\}}{G(x,y,z)}$$

for all $x,y,z \in X$ and $0 \le \alpha \le 1$

Then T has a unique fixed point and T is G-continuous at *u*.

2. MAIN RESULT

Theorem 2.1: Let X be a complete G-metric space. Suppose the mappings $f,g,h: X \to X$ satisfying for all $x,y,z \in X$

$$G(fx,gy,hz) \le \alpha \frac{\max.\{G^2(x,fx,gy), G^2(y,gy,hz), G^2(z,hz,fx)\}}{G(x,y,z)}$$

where $0 \le \alpha \le 1$ (2.1.1)

Then *f*, *g* and *h* have a unique fixed point in X.

Proof: Let x_0 be an arbitrary point in X, and let $x_n = fx_{n-1}$, $x_{n+1} = gx_n$, $x_{n+2} = hx_{n+1}$, for any n∈N. Assume $x_n \ne x_{n-1} \ne x_{n+1}$, for n ∈ N, then from (2.1.1) we have

$$G(x_n,x_{n+1},x_{n+2}) = G(fx_{n-1},gx_n,hx_{n+1})$$

$$\le \alpha \frac{\max.\left\{\begin{matrix} G^2(x_{n-1},fx_{n-1},gx_n), G^2(x_n,gx_n,hx_{n+1}), \\ G^2(x_{n+1},hx_{n+1},fx_{n-1}) \end{matrix}\right\}}{G(x_{n-1},x_n,x_{n+1})}$$

$$= \alpha \frac{\max.\left\{\begin{matrix} G^2(x_{n-1},x_n,x_{n+1}), G^2(x_n,x_{n+1},x_{n+2}), \\ G^2(x_{n+1},x_{n+2},x_n) \end{matrix}\right\}}{G(x_{n-1},x_n,x_{n+1})}$$

i.e. $G(x_n,x_{n+1},x_{n+2}) \le \alpha\ G(x_{n-1},x_n,x_{n+1})$

-------- (2. 1.2)

Similarly we can show that

$$G(x_{n-1},x_n,x_{n+1}) \le \alpha\ G(x_{n-2},x_{n-1},x_n)$$

---------------------- (2.1.3)

By induction we can write

$G(x_n,x_{n+1},x_{n+2}) \le \alpha^n\ G(x_0,x_1,x_2)$ for $n \ge 1$

------------------------ (2.1.4)

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} G(x_n,x_{n+1},x_{n+2}) = 0$$

Using G_3

$\lim_{n\to\infty} G(x_n,x_{n+1},x_{n+1}) = 0$ --------------- (2.1.5)

Now, we prove that $\{x_n\}$ is a G-Cauchy sequence.

On the contrary, we assume that $\{x_n\}$ is not a G-Cauchy sequence.

Therefore there exists $\in > 0$ for which we can find subsequence $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that,

$G(x_{n(k)},x_{m(k)},x_{m(k)}) \ge \in$ -----------(2.1.6)

Further corresponding to $m(k)$, we can choose $n(k)$ in such a way that the smallest integer with $n(k) > m(k)$ and satisfying (2.1.6)

Then $G(x_{n(k)-1},x_{m(k)},x_{m(k)}) < \in$ ------- (2.1.7)

$\therefore$ By using (2.1.6) and rectangular inequality we can write ,

$$\in \le G(x_{n(k)},x_{m(k)},x_{m(k)}) \le G(x_{n(k)},x_{n(k)-1},x_{n(k)-1}) + G(x_{n(k)-1},x_{m(k)},x_{m(k)})$$

$< G(x_{n(k)},x_{n(k)-1},x_{n(k)-1}) + \in$ ----------- (2.1.8)

Using $[G(x,y,z) = G(y,z,x) = G(z,x,y)]$ we can write

$$0 \le G(x_{n(k)},x_{n(k)-1},x_{n(k)-1}) = G(x_{n(k)-1},x_{n(k)-1},x_{n(k)})$$

Taking $k \to \infty$ and using (2.1.5) we get, $G(x_{n(k)},x_{n(k)-1},x_{n(k)-1}) \to 0$

Therefore (2.1.8) will reduces to

$\lim_{k\to\infty} G(x_{n(k)},x_{m(k)},x_{m(k)}) = \in$ ------------ (2.1.9)

By using rectangular inequality, we have,

$$G(x_{n(k)}, x_{m(k)}, x_{m(k)}) \leq G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{m(k)-1}, x_{m(k)}, x_{m(k)})$$

$$G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) \leq G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1})$$

Letting $k \to \infty$ in the above two inequalities and using (2.1.5) and (2.1.9)

$$\lim_{k\to\infty} G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) = \in \text{ -------} \qquad (2.1.10)$$

Setting $x = x_{n(k)-1}$, $y = x_{m(k)-1}$ and $z = x_{m(k)-1}$ in (2.1.1) we get a contradiction as $\in > 0$.

This shows that $\{x_n\}$ is a G-Cauchy sequence and since X is a G-complete space, $\{x_n\}$ is G-convergent to some $u \in X$

i. e. $\lim_{n\to\infty} G(x_n, x_n, u) = \lim_{n\to\infty} G(x_n, u, u) = 0$ (2.1.11)

We show that u is a fixed point of the mapping *f*

Let us assume that

$$G(x_{n+1}, x_{n+1}, fu) \leq \alpha \ G(fx_n, fx_n, fu)$$

By using (2.1.11) and preposition (1.7) we have

$$\lim_{n\to\infty} G(x_{n+1}, x_{n+1}, fu) = 0 \text{ -------------} \qquad (2.1.12.)$$

Again by using rectangle inequality condition of G-metric, we can write

$$G(u,u, fu) \leq G(u,u,x_{n+1}) + G(x_{n+1}, x_{n+1}, fu)$$

Taking limit as $n \to \infty$ in the above inequality, by using (2.1.11) and (2.1.12)

we have $G(u,u, fu) = 0$, this implies that $fu = u$

Hence *u* is fixed point of *f*.

Similarly we can show that u is fixed point of g and h .Hence *u* is common fixed point of mappings *f*, g and *h*.

Now, we show that *u* is unique fixed point of *f*, *g* and *h*.

For this let us assume that *v* is another fixed point of f, *g* and *h*.

$$\therefore \ G(u,u,v) = G(fu, gu, hv)$$

$$\leq \alpha \frac{\max.\{G^2(u, fu, gu), G^2(u, gu, hv), G^2(v, hv, fu)}{G(u,u,v)}$$

$$= \alpha \frac{\max.\{0, G^2(u,u,v), G^2(v,v,u)\}}{G(u,u,v)}$$

$\therefore \ G(u,u,v) \leq \alpha \ G(u,u,v)$, which is a contradiction.

Therefore $u = v$.

Hence *u* is unique common fixed point of *f, g* and *h*.

Example 2.2: Let $X = [0,\infty)$ and G be a mapping defined on X as

$$G(x,y,z) = \max.\{|x-y|, |y-z|, |x-z|\},$$

for all $x, y, z \in X$

Then G is a complete G - metric on X and (X,G) is a complete G -metric space.

Define $f, g, h : X \to X$ by $f(x) = \frac{x}{2}$,

$g(x) = 3x$ and $h(x) = x^2$, for all $x \in X$

Then by using condition (2.1.1) of Theorem 2.1 we get

$$G(fx, gy, hz) \leq \alpha \frac{\max.\{G^2(x, fx, gy), G^2(y, gy, hz), G^2(z, hz, fx)\}}{G(x,y,z)}$$

where $0 \leq \alpha \leq 1$,

for all $x, y, z \in X$.

Then 0 is the unique common fixed point of f, *g* and *h*.

3. CONCLUSION

In this study we obtained common fixed point results for three mappings in G-metric space under rational type contractive condition without using any stronger conditions like commutativity of the mappings. Thus our results improve, generalize and extend many recent results existing in the literature. An example is discussed to validate the main result of this paper.

4. FUTURE SCOPE

The above result may extend for six compatible mappings of G-matrix space.

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