Measures of Growth and Approximation of Analytic Functions in Several Complex Variables

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ABSTRACT
Winiarski [5] has studied the rates of decay of Lagrange interpolation error for entire functions in several complex variables. These results do not give any information about the rates of decay of above error when function is not necessarily entire. In this paper the authors have worked out this problem.

General Terms
Analytic Functions, Complex Variables.

Keywords
Holomorphically convex domain, Lagrange interpolation polynomial, Domain of holomorphy, Faber polynomials, Polydisc, Polyradius

1. INTRODUCTION
The purpose of this article is to extend the results of Winiarski [5] when the function is not necessarily entire, in several complex variables. Complex analysis differs dramatically between one and several variables. Many fundamental features change with space dimension from one to greater than one. For example, any domain on the complex plane is a domain of holomorphy but in this is not the case.

Now first let's discuss the domain of holomorphy. Let D be a domain in $C^n, n \geq 1.$

Definition 1.1
D is called a domain of holomorphy if there is a holomorphic function f defined on D that is singular at every boundary point, that is, f can not be extended holomorphically across any boundary point.
In single complex variable, every domain is a domain of holomorphy. But in higher-dimensional Euclidean spaces, the above concept in general need not hold. Take the following example in C^2. Let

$$D=\{(z_1,z_2)\in C^2:\frac{1}{2}<|z_2|<1\,\text{and}\,|z_1|<1\}U\{(z_1,z_2)\in C^2:|z_2|\leq\frac{1}{2}\text{and}\,|z_1|<\frac{1}{2}\}.$$

For any function f holomorphic on D, set

$$F(z)=\frac{1}{2\pi i}\int_\Gamma \frac{f(z_1,w)}{w-z_2}\,dw,$$

where $\Gamma=\{w\in C:|w|=3/4\}.$ Then it is easy to see that $F|_D=f$ on D and F is holomorphic on

D_1, where

$$D_1=\{(z_1,z_2):|z_1|<1\,\text{and}\,|z_2|<1\}.$$

This implies that D is not a domain of holomorphy since any holomorphic function f on D can be holomorphically extended to a strictly larger set via applying Cauchy integral formula in. Thus, it becomes fundamental to determine whether a given domain D is a domain of holomorphy or not. To avoid this difficulty, the authors have the following definition:

Definition 1.2
Let D be a domain in C^n. D is called holomorphically convex if $\widehat{K}_D$ is relatively compact in D for any compact subset K of D. Here $\widehat{K}_D=\{z\in D:|f(z)|\leq \sup_K|f|$ for all f holomorphic on D} is the convex hull of K and $D\setminus K$ is connected.
Since holomorphically convex hull of a compact set is always contained in the geometrically convex hull of this set, it follows that any convex domain is holomorphically convex. It turns out that holomorphic convexity is the right condition for characterizing domains of holomorphy in C^n. For clearity of the above concept the authors have the following Theorem due to H. Carten and P. Thullen [1].

2. THEOREM A
Let D be a domain in $C^n, n\geq 1.$ Then the following statements are equivalent

i. D is a domain of holomorphy.
ii. D is holomorphically convex.

Now the author assume that E be a holomorphically convex set in the space C^n of n complex variables $\tilde{z}=(z_1,...,z_n).$

Let $H(E)$ be the linear space of all functions analytic on E . Define a norm on $H(E)$ as,

$$\|f\|=\sup_{\tilde{z}\in E}\left\{|f(\tilde{z})|, f\in H(E)\right\}.$$

Let $\Im_\nu$ denote the set of all polynomials in $\tilde{z}$ of degree $\leq \nu$. For $f\in H(E),$ set

$$\lambda_{\nu,1}(f)\equiv\lambda_{\nu,1}(f,E)=\inf_{p\in\Im_\nu}\left\{\|f-p\|\right\}.$$

Let $k_{1l},\ldots,k_{nl}$, $l = 1.2,\ldots,\nu_*$, denote the sequence of all solutions in non-negative integers of the inequality

$$k_1 + \ldots + k_n \le \nu.$$

Let $p^{(\nu)} = \{p_1,\ldots,p_{\nu_*}\}$ be a system of ν_* points

$$p_i = \{z_{1i},\ldots,z_{ni}\},\ i = 1,\ldots,\nu_*,$$

such that the determinant

$$V(p^{(\nu)}) = \det\left[z_{1i}^{k_{1l}},\ldots,z_{ni}^{k_{nl}}\right],\ i,l = 1,\ldots,\nu_*,$$

is different from zero. Considering a new determinant $V_i\left(\tilde{z}, p^{(\nu)}\right)$, corresponding to the system of points $\left[p_1,\ldots,p_{i-1},z,p_{i+1},\ldots,p_{\nu_*}\right], z$ being an arbitrary point of C^n. Let

$$L^{(i)}\left(\tilde{z}, p^{(\nu)}\right) = \frac{V_i\left(\tilde{z}, p^{(\nu)}\right)}{V\left(p^{(\nu)}\right)},\ i = 1,\ldots,\nu_*.$$

The polynomial

$$L_\nu(\tilde{z}) = \sum_{i=1}^{\nu_*} f(p_i) L^{(i)}\left(\tilde{z}, p^{(\nu)}\right),\ \tilde{z} \in C^n,\ p^{(\nu)} \subset E,$$

is called the ν^{th} Lagrange interpolation polynomial for f with nodes $p^{(\nu)}$.

Further, put

$$\lambda_{\nu,2}(f) \equiv \lambda_{\nu,2}(f,E) \equiv \| L_\nu - L_{\nu-1} \|,$$

$$\lambda_{\nu,3}(f) \equiv \lambda_{\nu,3}(f,E) \equiv \| f - L_\nu \|.$$

Winiarski [5] has studied the rates of decay of $\lambda_{\nu,j}(f), j = 1,2,3,$ for entire functions. His results do not give any information about the rates of decay of $\lambda_{\nu,j}(f), j = 1,2,3,$ when $f(\tilde{z}) \in H(E)$ is not necessarily entire. In the present paper an attempt has been made to solve this problem.

Now first define the extremal function [2]

$$\phi(\tilde{z}) = \phi(\tilde{z},E) = \lim_{\nu\to\infty}\left\{\sup\left\{\left| p(\tilde{z})\right|^{1/\nu} : p \in A_\nu(E)\right\}\right\}, \tilde{z} \in C^n$$

and $A_\nu(E)$ is the set of all polynomials p of degree $\le \nu$ such that $\| p \|_E \le 1.$

It can be seen from the definition that $\phi(\tilde{z}) \ge 1$ for $\tilde{z} \in C^n$ and $\phi(\tilde{z}) = 1$ for $\tilde{z} \in E.$

Let $\phi(\tilde{z},E)$ be locally bounded and

$$E_{\bar{r}} = \left\{\tilde{z} \in C^n : \phi(\tilde{z},E) = \bar{r}\right\}, \bar{r} > \bar{1}, \bar{r} \equiv (r,\ldots,r), \bar{1} \equiv (1,\ldots,1).$$

Then $E_{\bar{r}}$ is an analytic Jordan curve. Let $D_{\bar{r}}$ be the convex domain with the boundary $E_{\bar{r}}$. Then $E \subset D_{\bar{r}}$ for each $\bar{r}\,(\bar{1} < \bar{r} < \infty)$ and $E_{\bar{r}} \subset D_{\bar{r}'}$, for $\bar{r}' > \bar{r}$. Since through an arbitrary point $\tilde{z}_o \notin E$ there passes one and only one curve $E_{\bar{r}}\,(\bar{1} < \bar{r} < \infty)$, it follows that for each $f \in H(E)$ there exists an unique $\bar{R} \equiv \bar{R}(f)\ (\bar{1} < \bar{R} \le \infty)$ such that f can be extended analytically to $D_{\bar{r}}$ for each $\bar{r} \le \bar{R}$ but for no $\bar{r} > \bar{R}.$ The authors call $D_{\bar{R}}$ as the "domain of holomorphy" for f and denote the class of those $f \in H(E)$ which have the domain of holomorphy $D_{\bar{R}}$ by $H(E,\bar{R}).$

Now define the growth parameters for a function $f \in H(E,\bar{R})$ as follows:

A function $f \in H(E,\bar{R}), \bar{1} < \bar{R} < \infty,$ will be said to be of order ρ [7] if

$$\rho = \limsup_{t\to 1} \frac{\log^+ \log^+ M_{E_{\bar{R}}}(t,f)}{-\log(1-t)}$$

Where $M(\bar{r}, f) = \max\limits_{\substack{|z_i| = r_i \\ i=1,\ldots,n}} | f(\tilde{z}) |$

and

$$M_{E_{\bar{R}}}(t,f) = \max_{\bar{r} \in tE_{\bar{R}}} M(\bar{r}, f),\ 0 \le r_i < R,\ 0 < t < 1.$$

In case $0 < \rho < \infty$, the type T of f is defined as

$$T = \limsup_{t\to 1} \frac{\log^+ M_{E_{\bar{R}}}(t,f)}{(1-t)^{-\rho}}.$$

Here $\log^+ x = \max(0, \log x).$

In this the authors prove the following lemmas that are needed in the sequel.

Let

$$G^n\left(\tilde{z}^o, \bar{R}\right) = \left\{\tilde{z} = (z_1,\ldots,z_n) : \left|z_i - z_i^o\right| < R_i,\ i = 1,\ldots,n\right\}$$

be a polydisc in C^n with centre $\tilde{z}^o = \left(z_i^o,\ldots,z_n^o\right)$ and polyradius $\bar{R} = \left(R_i,\ldots,R_n\right)$, where z_i^o are complex numbers and R_i are fixed real numbers for $i = 1,\ldots,n.$ For the sake of simplicity, the author assume throughout that $\tilde{z}^o = \tilde{0},\ \bar{R} \equiv (R,\ldots,R)$ and denote $G \equiv G^n(\tilde{0},\bar{R}).$

A function f , analytic in $G^n(0,\bar{R}),$ is uniquely expressed as an absolutely convergent power series

$$f(\tilde{z}) = \sum_{\|\bar{k}\|=0}^{\infty} b_{\bar{k}} \tilde{z}^{\bar{k}} \quad \tilde{z} \in G^n(0,\bar{R}),$$

where, $\bar{k} = (k_1,...,k_n)$ is n-tuple of nonnegative integers, $\tilde{z}^{\bar{k}} = z_{1j}^{k_1},...,z_n^{k_n}$ and $\|\bar{k}\| = k_1 + ... + k_n$. With $\bar{r} = (r_1,...,r_n),\ 0 \le r_i < R,\ i = 1,...,n,$ let

$$M(\bar{r}, f) = \max_{|z_i| \le r_i} |f(\tilde{z})| = \max_{|z_i| = r_i} |f(\tilde{z})|,$$

be the maximum modulus of f on the closed polydisc $\bar{G}^n(\tilde{0}, \bar{R})$. Set

$$M_G(t, f) = \max_{\bar{r} \in tG} M(\bar{r}, f),\ 0 < t < 1,$$

Where

$$G \equiv |G| = \{(r_1,...,r_n) \in R^n : 0 \le r_i < R_i,\ i = 1,...,n\}.$$

Define $G_{\bar{R}}$ – order $\rho_{G,\bar{R}}$ of f as

$$\rho_{G,\bar{R}} = \limsup_{t \to 1} \frac{\log^+ \log^+ M_G(t, f)}{-\log(1-t)}.$$

If $0 < \rho_{G,\bar{R}} < \infty$, the G-type $T_{G,\bar{R}}$ of f is defined as

$$T_{G,\bar{R}} = \limsup_{t \to 1} \frac{\log^+ M_G(t, f)}{(1-t)^{-\rho_{G,\bar{R}}}}.$$

Now first the author obtain coefficient characterizations of $G_{\bar{R}}$ – order and $G_{\bar{R}}$ – type of a function analytic in $G^n(0,\bar{R})$.

Lemma 2.1

Let $f(\tilde{z}) = \sum_{\|\bar{k}\|=0}^{\infty} b_{\bar{k}} \tilde{z}^{\bar{k}}$ be analytic in the polydisc $G^n(0,\bar{R})$ and have $G_{\bar{R}}$ – order $\rho_{G,\bar{R}}$, $0 \le \rho_{G,\bar{R}} \le \infty$. Then

$$\frac{\rho_{G,\bar{R}}}{\rho_{G,\bar{R}} + 1} = \limsup_{\|\bar{k}\| \to \infty} \left\{ \frac{\log^+ \log^+ |b_{\bar{k}}| \bar{R}^{\bar{k}}}{\log \|\bar{k}\|} \right\}.$$

Lemma 2.2

Let $f(\tilde{z}) = \sum_{\|\bar{k}\|=0}^{\infty} b_{\bar{k}} \tilde{z}^{\bar{k}}$ be analytic in the polydisc $G^n(0,\bar{R})$ and $G_{\bar{R}}$ – order $\rho_{G,\bar{R}}$, $(0 < \rho_{G,\bar{R}} < \infty)$ and $G_{\bar{R}}$ – typeT$_{G,\bar{R}}$ $(0 \le T_{G,\bar{R}} \le \infty)$. Then

$$\frac{(\rho_{G,\bar{R}} + 1)^{\rho_{G,\bar{R}}+1}}{\rho_{G,\bar{R}}^{\ \rho_{G,\bar{R}}}} T_{G,\bar{R}} = \limsup_{\|\bar{k}\| \to \infty} \left\{ \frac{\left(\log^+ |b_{\bar{k}}| \bar{R}^{\bar{k}}\right)^{\rho_{G,\bar{R}}+1}}{\|\bar{k}\|^{\rho_{G,\bar{R}}}} \right\}.$$

Proofs of Lemma 2.1 and 2.2

A function g, analytic in $G^n(0,\bar{1})$ is uniquely expressed an absolutely convergent power series [4]

$$g(\tilde{z}) = \sum_{\|\bar{k}\|=0}^{\infty} b_{\bar{k}} \tilde{z}^{\bar{k}},\ \tilde{z} \in G^n(0,\bar{1}).$$

Consider the function $f(\tilde{z}) = g(\bar{R}\tilde{z}),$ where $\bar{R}\tilde{z} = (R_1 z_1,...,R_n z_n),$ is analytic in $G^n(0,\bar{R})$. It can be easily seen that order and type of f and g are equal. Hence using Theorems 5.2.1 and 5.2.2 of Juneja and Kapoor [6], the authors get the required results.

Set

$$E_{\bar{r}}^* = \{\tilde{z} : \phi(\tilde{z}, E) = \bar{r}\} \bar{r} > \bar{\beta}, \bar{\beta} > \bar{1} \text{ and } D_{\bar{r}}^*$$

is the convex domain interior to $E_{\bar{r}}^*$.

Lemma 2.3

Let $f(\tilde{z})$ be analytic in $D_{\bar{r}_o}^*, \bar{r}_o > \bar{\beta}$. Then there exists a polynomial $Q_\nu \in \Im_\nu$ such that

$$|f(\tilde{z}) - Q_\nu(\tilde{z})| \le K M_{E_{\bar{r}}^*}(\bar{r}, f) (\bar{\beta}/\bar{r})^\nu,\ \tilde{z} \in E,$$

for all $\bar{r}\ (< \bar{r}_o)$ sufficiently close to $\bar{r}_o$. Here K is a constant depending on the set E and $\bar{r}_o$ but is independent of ν and $\bar{r}_o$.

Proof. It can be seen as in the case of single complex variable [3, p.109], [5, p. 138] that inside $D_{\bar{r}_o}^*$, $f(\tilde{z})$ can be expressed as

$$f(\tilde{z}) = \sum_{\|\bar{k}\|=0}^{\infty} b_{\bar{k}} h_{\bar{k}}(\tilde{z}) \tag{2.1}$$

where $\{h_{\bar{k}}\}_{\|\bar{k}\|=0}^{\infty}$ is the sequence of Faber polynomials for E and the righ333t hand side of above series is uniformly convergent on every convex subsets of $D_{\bar{r}_o}^*$.

It is known that the analogues of Cauchy inequality in one variable case for $f(\tilde{z})$ is given by

$$|b_{\bar{k}}| \le \frac{M_{E_{\bar{r}}^*}(\bar{r}, f)}{\bar{r}^{\bar{k}}} \quad \text{for} \quad \bar{\beta} < \bar{r} < \bar{r}_o. \tag{2.2}$$

It is also known [5, p.137] as in the case of one variable that

$$\left| h_{\bar{k}}(\tilde{z}) \right| \le 2\,\bar{\beta}^{\bar{k}} \quad \text{for} \quad \tilde{z} \in E, \qquad (2.3)$$

Where $\bar{\beta}^{\bar{k}} = \beta_1^{k_1},...,\beta_n^{k_n}$.

Let $Q_\nu(z) = \sum_{\|\bar{k}\|=0}^{\nu} b_{\bar{k}}\, h_{\bar{k}}(\tilde{z})$ and

$$\left| f(\tilde{z}) - Q_\nu(\tilde{z}) \right| \le \sum_{\|\bar{k}\|=\nu+1}^{\infty} \left| b_{\bar{k}} \right| \left| h_{\bar{k}}(\tilde{z}) \right|.$$

In view of (2.2) and (2.3) it gives

$$\le 2M_{E_{\bar{r}}^*}(\bar{r}, f) \sum_{\|\bar{k}\|=\nu+1} \left(\bar{\beta}/\bar{r}\right)^{\bar{k}}, \quad \bar{\beta} < \bar{r} < \bar{r}_o,$$

$$\le \frac{2M_{E_{\bar{r}}^*}(\bar{r}, f)(\bar{\beta}/\bar{r})^{\nu+1}}{(1-(\bar{\beta}/\bar{r}))^2}, \quad \tilde{z} \in E.$$

Now taking $\bar{r} \ge ((\bar{\beta} + \bar{r}_o)/2)$ if $\bar{r}_o < \infty$ and $\bar{r} \ge 2\bar{\beta}$ if $\bar{r}_o = \infty,$ the authors get the required proof

Lemma2.4
Let

$f \in H(E, \bar{\bar{R}}),\ \bar{R} > \bar{1}$. Then

$$\lambda_{\nu,1}(f) \le K(\nu+1) M_{E_{\bar{r}}^*}(\bar{r}, f)(\bar{1}/\bar{r})^{\nu}, \quad \nu = 0,1,2,...$$

for all $\bar{r}\ (< \bar{R})$ sufficiently close to $\bar{R}$. Here K is a constant depending on E and $\bar{R}$ but independent of ν and $\bar{r}$.

Proof. In the consequence of Lemma 2.3 and definition of $E_{\nu,1}(f)$ the proof is immediate.

Lemma 2.5

Let $f \in H(E),$ then

(i) $\lambda_{\nu,1}(f, E) \le \lambda_{\nu,3}(f, E) \le (1+\nu_*)\lambda_{\nu,1}(f, E),$ for $\nu = 1,2,...,$

(ii) $\lambda_{\nu,2}(f, E) \le 2(1+\nu_*)\lambda_{\nu-1,1}(f, E).$

Proof. (i) Let p_ν be a polynomial of degree $\le \nu$ and let P_ν be the ν-th Lagrange interpolation polynomial for the function $g^*(\tilde{z}) = f(\tilde{z}) - p_\nu(\tilde{z}).$

Since

$$L_\nu(\tilde{z}) = p_\nu(\tilde{z}) + P_\nu(\tilde{z}) \quad \text{for } \tilde{z} \in C^n.$$

Therefore

$$\lambda_{\nu,1}(f, E) \le \| f - L_\nu \|_E \le \| f - p_\nu \|_E + \| P_\nu \|_E$$

$$\le \| f - p_\nu \|_E \left(1 + \sum_{j=1}^{\nu_*} \| L^{(j)} \| \right) \le \| f - p_\nu \|_E (1+\nu_*),$$

which completes the proof of (i) part. The part (ii) can be proved similarly.

3. MAIN RESULTS

Theorem 3.1

Let $f \in H(E),$ then $f \in H(E, \bar{R}), \bar{R} > \bar{1},$ if and only if, $\limsup_{\nu\to\infty} \left(\lambda_{\nu,j}(f)\right)^{1/\nu} = \bar{1}/\bar{R}$, $j = 1,2,3.$

Proof. Let $f \in H(E, \bar{R}).$ Then by Lemma 2.4, the authors get

$$\limsup_{\nu\to\infty} \left(\lambda_{\nu,1}(f, E)\right)^{1/\nu} \le \bar{1}/\bar{r}$$

for all $\bar{r}\ (< \bar{R})$ sufficiently close to $\bar{R}$ and so

$$\limsup_{\nu\to\infty} \left(\lambda_{\nu,1}(f, E)\right)^{1/\nu} \le \bar{1}/\bar{R}.$$

On the other hand, let $\phi(\tilde{z}, w)$ of $(n+1)$ complex variables $z_1,..., z_n, w$ defined by

$$\phi(\tilde{z}, w) = f(w\tilde{z}) = \sum_{\|\bar{k}\|=0}^{\infty} b_{\bar{k}} w^{\|k\|}\, \bar{z}^{\bar{k}}, \quad |w| < R.$$

Set

$$P_\nu(\tilde{z}) = \sum_{\|\bar{k}\|=\nu} b_{\bar{k}}\, \bar{z}^{\bar{k}}.$$

Then

$$\phi(\tilde{z}, w) = \sum_{\nu=0}^{\infty} P_\nu(\tilde{z})\, w^\nu$$

is analytic function of w in $G^n(0, \bar{R}).$ Therefore

$$f(\tilde{z}) = P_0(\tilde{z}) + \sum_{\nu=1}^{\infty} \left(P_{\nu+1}(\tilde{z}) - P_\nu(\tilde{z}) \right) \qquad (3.1)$$

holds in $D_{\bar{R}}$ and the right hand side of above series converges uniformly on every compact subsets of $D_{\bar{R}}$.

Since

$$\| P_{\nu+1}(\tilde{z}) - P_\nu(\tilde{z}) \|_E \le \lambda_{\nu+1}(f, E) + \lambda_\nu(f, E) \le 2\lambda_\nu(f, E).$$

Now using the property of extremal function [2], the authors get

$$\left| P_{\nu+1}(\tilde{z}) - P_\nu(\tilde{z}) \right| \le 2\lambda_{\nu,1}(f, E)\,\bar{r}^{(\nu+1)}, \quad \tilde{z} \in E_{\bar{r}}, \ \bar{r} > \bar{1}. \qquad (3.2)$$

It has been seen that if

$\limsup_{\nu\to\infty}\left(\lambda_{\nu,1}(f,E)\right)^{1/\nu} < \overline{1}/\overline{R},$ then the series on the right hand side of (3.1) converges uniformly on every compact subsets of $D_{\overline{R}'}$, for some $\overline{R}' > \overline{R}$, which is a contradiction. Hence $\limsup_{\nu\to\infty}\left(\lambda_{\nu,1}(f,E)\right)^{1/\nu} = \overline{1}/\overline{R}.$

This prove the necessary part of the theorem for $j=1.$ In view of Lemma 2.5, it can also be proved for $j=2,3.$ Sufficiency part can be proved in a similar manner.

Theorem 3.2

Let $f \in H(E,\overline{R}), \overline{1} < \overline{R} < \infty,$ be of order ρ. Then

$$\frac{\rho}{\rho+1} = \limsup_{\nu\to\infty} \frac{\log^+\log^+ \lambda_{\nu,j}(f,E)\overline{R}^{\nu}}{\log \nu}. \quad (3.3)$$

Proof. First prove the theorem for j = 1, let

$$\limsup_{\nu\to\infty} \frac{\log^+\log^+ \lambda_{\nu,j}(f,E)\overline{R}^{\nu}}{\log \nu} = \alpha.$$

Obviously $0 \le \alpha \le \infty.$ First assume that $0 < \alpha < \infty$ and $0 < \alpha' < \alpha < \infty.$ Then by the definition of α there exists a sequence $\{\nu_{\bar{k}}\}$ of positive integers tending to ∞ such that

$$\log \lambda_{\nu_{\bar{k}},1}(f,E)\,\overline{R}^{\nu_{\bar{k}}} > \nu_{\bar{k}}^{\alpha'}.$$

In view of Lemma 2.4, the authors get

$$\log M_{E_{\bar{r}}^*}(\bar{r},f) \ge \nu_{\bar{k}}^{\alpha'} + \nu_{\bar{k}}\log(\bar{r}/\overline{R}) - \log(\nu_{\bar{k}}+1) - \log K \quad (3.4)$$

for the sequence $\{\nu_{\bar{k}}\}$ and all $\bar{r}\,(<\overline{R})$ sufficiently close to $\overline{R}.$ Let $\{\bar{r}_{\bar{k}}\}$ be a sequence defined by

$$\nu_{\bar{k}} = \left\{\frac{1}{\alpha'}\log\left(\overline{R}/\bar{r}_{\bar{k}}\right)\right\}^{1/\alpha'-1},$$

then $\bar{r}_{\bar{k}} \to \overline{R}$ as $\|\bar{k}\| \to \infty.$ Thus, using (3.4), for all sufficiently large values of $\|\bar{k}\|$ the authors have

$$\log M_{E_{\bar{r}}^*}\left(\bar{r}_{\bar{k}},f\right) \ge \left(\frac{1}{\alpha'}\log\left(\overline{R}/\bar{r}_{\bar{k}}\right)\right)^{\alpha'/\alpha'-1} - \left(\frac{1}{\alpha'}\log\left(\overline{R}/\bar{r}_{\bar{k}}\right)\right)^{1/\alpha'-1}\log\left(\overline{R}/\bar{r}_{\bar{k}}\right) + O(1)$$

$$= \frac{(1-\alpha')}{\alpha'^{(\alpha'/\alpha'-1)}}\left(\log\left(\overline{R}/\bar{r}_{\bar{k}}\right)\right)^{\alpha'/\alpha'-1} + O(1)$$

Or

$$\log^+\log^+ M_{E_{\bar{r}}^*}\left(\bar{r}_{\bar{k}},f\right) \ge \frac{\alpha'}{1-\alpha'}\left(-\log\log\left(\overline{R}/\bar{r}_{\bar{k}}\right)\right) + O(1).$$

Since $-\log\left(1-\bar{r}_{\bar{k}}/\overline{R}\right) \sim -\log\log\left(\overline{R}/\bar{r}_{\bar{k}}\right),$

Therefore,

$$\log^+\log^+ M_{E_{\bar{r}}^*}\left(\bar{r}_{\bar{k}},f\right) \ge \frac{\alpha'}{1-\alpha'}\left(-\log\left(1-\bar{r}_{\bar{k}}/\overline{R}\right)\right) + O(1) \text{ as } \|\bar{k}\| \to \infty.$$

Using $\max f(\tilde{\mathfrak{I}}) \le \max$

$$M_{E_{\bar{r}}^*}(\bar{r}_{\bar{k}}^*, f) = M_{E_{\bar{r}}^*}(r,f), o < t < 1,$$

$$|\mathfrak{I}_i| = t\,r_{k_i} \qquad \bar{r}_{\bar{k}}^* \in t\,E_{\bar{r}}^*$$

$$1 \le i \le n$$

in above inequality, the authors get

$$\limsup_{t\to 1} \frac{\log^+\log^+ M_{E_{\bar{r}}^*}(t,f)}{-\log(1-t)} \ge \frac{\alpha'}{1-\alpha'}.$$

Since $\alpha'(<\alpha)$ is arbitrary, therefore

$$\rho \ge \frac{\alpha}{1-\alpha}$$

$$\frac{\rho}{\rho+1} \ge \alpha. \quad (3.5)$$

Now, (3.1) gives that

$$f(\tilde{z}) = P_o(\tilde{z}) + \sum_{\nu=1}^{\infty}\left(P_{\nu+1}(\tilde{z}) - P_\nu(\tilde{z})\right)$$

holds in $D_{\bar{r}}$. Thus in view of (3.2), the authors get

$$|f(\tilde{z})| \le |P_o(\tilde{z})| + \sum_{\nu=1}^{\infty}|P_{\nu+1}(\tilde{z}) - P_\nu(\tilde{z})|$$

$$\le \overline{K} + 2\sum_{\nu=1}^{\infty}\lambda_{\nu,1}(f,E)\,\bar{r}^{\nu+1} \quad (3.6)$$

for $\tilde{z} \in E_{\bar{r}}, \overline{1} < \bar{r} < \overline{R}.$ Here $\overline{K}$ is constant independent of $\bar{r}.$ Now, (3.6) gives

$$M_{E_{\bar{r}}}(\bar{r},f) \le \overline{K} + 2M_{E_{\bar{r}}}(\bar{r},h) \quad (3.7)$$

where $h(\tilde{z}) = \sum_{\nu=o}^{\infty}\lambda_{\nu,1}(f,E)\,\tilde{z}^{\nu+1}.$

By Theorem 3.1, $h(\tilde{z})$ is analytic in $G^n(0,\overline{R}).$ Using (3.7) with Lemma 2.1 for $h(\tilde{z}),$ the authors get

$$\frac{\rho}{\rho+1} \le \alpha. \quad (3.8)$$

Hence the proof is complete for $0 < \alpha < \infty$ in view of (3.5) and (3.8) together with Lemma 2.5. For $\alpha = 0$ or $\infty,$ the proof is trivially true.

Finally,

Let $\tilde{f}(\tilde{z}) = \lim_{\nu\to\infty} L_\nu(\tilde{z}),\ \tilde{z} \in C^n.$

It has been noticed that the type of $\tilde{f}$ cannot be characterized in terms of the Chebyshev best approximation to f on E by polynomials of degree $\le \nu$ with respect to all variables. So it is convenient to consider the measures $\lambda_{k_i}(f, E_i),\ k_i = (k_1,...,k_n)$ of the Chebyshev best approximation to f on E_i by polynomials of degree $\le k_i$ with respect to ith variable, $i = 1,...,n,$ where E_i is a bounded closed set in the complex $z_i -$ plane. Hence the authors have obtained the characterization of type of $\tilde{f}$ in terms of $\lambda_{k_i}(f, E_i)$ using the concept of partial order and partial type introduced by Juneja and Kapoor [2, pp.273].

Now let's prove the following theorem:

Theorem 3.3

Let $f \in H(E, \bar{R}), \bar{1} < R_i < \infty,$ and if $\rho_i = (\rho_1,...,\rho_n) > (0,...,0),$ $T_i = (T_1,...,T_n) > (0,...,0)$ are partial order and partial type of f , respectively, then

$$\limsup_{\min\{k_i\}\to\infty} \|\bar{k}\| \sqrt{\log \lambda_{k_{i,j}}(f, E_i) R_i^{k_i} \left(\frac{\rho_i^{\rho_i}}{(\rho_i+1)^{\rho_i+1} k_i^{\rho_i} T_i}\right)^{k_i/(\rho_i+1)}} = 1,$$

where $\left(\frac{\rho_i^{\rho_i}}{(\rho_i+1)^{\rho_i+1} k_i^{\rho_i} T_i}\right)^{k_i/(\rho_i+1)}$

=

$$\left(\frac{\rho_1^{\rho_1}}{(\rho_1+1)^{\rho_1+1} k_1^{\rho_1} T_1}\right)^{k_1/\rho_1+1}, ..., \left(\frac{\rho_n^{\rho_n}}{(\rho_n+1)^{\rho_n+1} k_n^{\rho_n} T_n}\right)^{k_n/\rho_n+1}$$

and $\|\bar{k}\| = k_1 + \ldots + k_n$.

Proof. Let f be of partial type T_i , then for given $\varepsilon_i = (\varepsilon_1,...,\varepsilon_n) > (0,...,0)$ there exists $r_i = r_i(\varepsilon_i),\ r_i < \bar{r} < R_i$ such that

$$\limsup_{t\to 1} \frac{\log^+ M_G(t, f)}{(1-t)^{-\rho_i}} = T_{i,G,R_i} \qquad (3.9)$$

for all $r_i (< R_i)$ sufficiently close to R_i . Further (3.9) can be written as

$$\limsup_{r_i\to R_i} \frac{\log^+ M_{E_{\bar{r}}^*}(\bar{r}, f)}{((R_i - r_i)/R_i)^{-\rho_i}} = T_{i,E_{\bar{r}}^*} \equiv T_i$$

Or

$$\log^+ M_{E_{\bar{r}}^*}(\bar{r}, f) \le (T_i + \varepsilon_i)(R_i / R_i - r_i)^{\rho_i}. \qquad (3.10)$$

In view of Lemma 2.4 with respect to ith variable keeping other fix, (3.10) gives

$$\log^+ M_{E_{\bar{r}}^*}(\bar{r}, f) \ge \log E_{k_{i,1}}(f, E_i) + k_i \log r_i - \log(k_i+1) - \log K_i$$

$$\log \lambda_{k_{i,1}}(f, E_i) \le (T_i + \varepsilon_i)(R_i / R_i - r_i)^{\rho_i} - k_i \log r_i + \log(k_i+1) + \log K_i. \qquad (3.11)$$

Now, choosing a sequence $\{r_{m_i}\}$ as

$$(R_i / R_i - r_{m_i}) = \left(\frac{k_i}{(T_i + \varepsilon_i)\rho_i}\right)^{1/(\rho_i+1)}. \qquad (3.12)$$

Clearly $r_{m_i} \to R_i$ as $m_i \to \infty.$ Putting (3.12) in (3.11), gives the result

$$\log \lambda_{k_{i,1}}(f, E_i) R_i^{k_i} \le (T_i + \varepsilon_i)\left(\frac{k_i}{(T_i+\varepsilon_i)\rho_i}\right)^{\rho_i/(\rho_i+1)} (1 + \rho_i + O(1))$$

$$= (T_i + \varepsilon_i)^{1/(\rho_i+1)} \left(\frac{k_i}{\rho_i}\right)^{\rho_i/(\rho_i+1)} (1 + \rho_i + O(1))$$

Or

$$\limsup_{\min\{k_i\}\to\infty} \|\bar{k}\| \sqrt{\log \lambda_{k_{i,j}}(f, E_i) R_i^{k_i} \left(\frac{\rho_i^{\rho_i}}{(\rho_i+1)^{\rho_i+1} k_i^{\rho_i} T_i}\right)^{k_i/(\rho_i+1)}} \le 1.$$

The reverse inequality can be proved by applying Lemma 2.2 to the function $h(\tilde{z})$ with respect to ith variable. This completes the proof for $j = 1.$ In view of Lemma 2.5, the theorem can be proved for $j = 2,3.$

4. CONCLUSION

Using above results one can get information about the rate of decay of the error discussed by Winiarski [5] and Markushevich [3]. Moreover, these results can be used to limit the results when the function is not necessarily analytic.

5. REFERENCES

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